Title
Energy and Entropy in Quantum Field Theories

Author
Levine, Adam

Publication Date
2019

Peer reviewed|Thesis/dissertation

Energy and Entropy in Quantum Field Theories

by

Adam Levine

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Raphael Bousso, Chair
Professor Richard Borcherds
Professor Yasunori Nomura

Summer 2019

Energy and Entropy in Quantum Field Theories

Abstract

Energy and Entropy in Quantum Field Theories

by

Adam Levine

Doctor of Philosophy in Physics

University of California, Berkeley

Professor Raphael Bousso, Chair

Energy conditions play an important role in constraining the dynamics of quantum field theories as well as gravitational theories. For example, in semi-classical gravity, the achronal averaged null energy condition (AANEC) can be used to prove that it is always slower to traverse through a wormhole than to travel around via its exterior. Such conditions prevent causality violations that would lead to paradoxes. Recent advances have been made in proving previously conjectured energy conditions directly in quantum field theory (QFT) as well as in uncovering new ones.

This thesis will be an exploration of various energy inequalities in conformal field theories (CFTs) as well as semi-classical quantum gravity. At the core of this work lies the recently proved quantum null energy condition (QNEC). The QNEC bounds the null energy flowing past a point by a certain second shape derivative of entanglement entropy. We will demonstrate that the QNEC represents a deep connection between causality, energy and entanglement in quantum field theories. We explore this connection first in the context of holographic CFTs. Calculations in holographic theories will lead us to conjecture that the so-called diagonal QNEC is saturated in all interacting QFTs. We will then provide further, independent evidence that this conjecture holds for all CFTs with a twist gap by explicitly calculating shape derivatives of entanglement entropy using defect CFT techniques.

To my grandfathers, Bob and Lenny, who I know would have enjoyed learning about quantum entanglement.

To Paul and Barbara O'Rourke.

Contents

Acknowledgments

This work would have been impossible without the support and guidance of many people. Chief among them is my advisor, Raphael Bousso, who, despite not yet being my collaborator, has been extremely supportive of me in every aspect of this work. I could not have asked for a better advisor.

I also benefited immensely from the patient guidance of and collaboration with Stefan Leichenauer. I'm not sure exactly what I would have done without him, but I know it would have been significantly worse.

My collaboration with Tom Faulkner, which is detailed in the final chapter of this work, had a large impact on me. Working with Tom was a humbling and inspiring experience. I am extremely grateful for his patience with us graduate students at Berkeley, especially as our project wore on into its second year!

I benefited from many influential conversations with Aron Wall, who taught me a ton throughout graduate school.

I am also grateful to the Kavli Institute for Theoretical Physics, where I spent a semester as a graduate fellow. While at KITP, I talked frequently with Don Marolf who was extremely supportive, patient and kind to me.

To everyone at BCTP - Arvin, Chris, Ven, Illan, Pratik, Vincent, Jason, Zach, Mudassir - thanks for making these past few years so fun. I hope we will continue yelling at each other over physics for many years to come.

This work benefited from the support of the Defense Department through the National Defense Science and Engineering Graduate (NDSEG) fellowship.

Chapter 1

Introduction

The past two decades of research into quantum gravity has seen a rapid convergence of techniques from quantum information, string theory and quantum field theory. Much of this research has been anchored on the fundamental result of Maldacena, who found the first concrete example of the so-called "holographic principle" in string theory [113]. The holographic principle predicts that the information content of gravitating systems - such as a black hole - can be encoded in a theory of one lower dimension [132, 84].

Maldacena found a precise realization of this principle in the so-called AdS/CFT duality, which equates string theory on anti- de Sitter space (AdS) in $d+1$ spacetime dimensions with a special non-gravitational theory - a conformal field theory (CFT) - in d spacetime dimensions.

Although AdS/CFT provided further evidence for the holographic principle, the idea of a holographic universe arose out of a famous formula, due originally to Jacob Bekenstein, stating that black holes have entropy which scales not with their space-time volume but rather their horizon area [11]. The relationship between entropy and horizon area precisely takes the form

$$S_{BH} = \frac{A_{\text{horizon}}}{4G\hbar} \tag{1.0.1}$$

where $G\hbar = \ell_{\text{Planck}}^{d-2}$ is the Planck area for a spacetime of dimension d.

Although this fundamental formula was originally conjectured using simple thought experiments in the 1970s, it was not given a precise, UV realization until 2006, when Ryu & Takayanagi first found that the entropy for some sub-region $\mathcal{R}$ in a holographic CFT is in fact given by the area over $4G\hbar$ of a specific surface in the dual AdS spacetime [126, 125]. This co-dimension 2 surface, now called the Ryu-Takayanagi (RT) surface, is found by extremizing over all surfaces anchored and homologous to the boundary region $\mathcal{R}$.

This result opened the floodgates for research examining the connection between geometry and entanglement. To cherry pick a few important examples, Mark van Raamsdonk's work in [121] suggested that space-time connectivity should be related to entanglement. Furthermore, the work of Maldacena and Susskind conjectured that [112] entanglement between the exterior and interior Hawking modes of an evaporating black hole should holographically generate a geometric, wormhole-like connection. These fascinating ideas together with Einstein's equations suggest that if geometry (and therefore spacetime curvature) are related to entanglement there should be a corresponding connection between entanglement and energy density. This latter connection will be the main the subject of this thesis.

1.1 Energy conditions in gravity and quantum field theory

To understand the various connections between energy and entanglement, it is instructive to first review various constraints on energy in both semi-classical quantum gravity and quantum field theory. Einstein's equations can always be solved trivially given a metric by computing the Einstein tensor and then declaring that this gives you the stress tensor. Of course, such a procedure for solving Einstein's equations does not tell you whether the solution is physical.

For this question, one must consult energy conditions, which constrain the source term in Einstein's equations. In classical gravity, the weakest energy condition which is manifestly true is the null energy condition (NEC). The NEC states that at every point in the spacetime

$$T_{kk}(x) \geq 0 \tag{1.1.1}$$

where k^a is a null vector in the tangent space at x and where $T_{\mu\nu}$ is the stress tensor of the field theory coupled to gravity. For example, in free, scalar field theory, the null-null component of the stress tensor is

$$T_{kk} \sim \nabla_k \phi \nabla_k \phi \tag{1.1.2}$$

which is manifestly positive. In quantum field theory, this positivity can break due to quantum fluctuations. The canonical example of such a NEC-violating state is the Casimir vacuum. A more relevant example for our interests is that of the evaporating black hole, which has negative null energy outside the horizon [34] or the more recent examples of [65].

Thankfully, there is a weaker, less-local condition which appears to be more broadly true: the achronal averaged null energy condition (ANEC). The ANEC states that for every complete, achronal null geodesic γ, then the integral

$$\int_\gamma \langle T_{kk}(\lambda) \rangle \, d\lambda \tag{1.1.3}$$

is non-negative. Here λ is an affine parameter for γ.

The achronality condition is quite important, as counter examples to the ANEC without this condition can easily be constructed. Recently, an important counter example of this type was discovered in [65] in the context of AdS/CFT. We will return to this construction in Chapter 2.

One should note that the existence of an achronal, complete null geodesic is highly non-generic, since any positive energy will cause the curve to become chronal. This means that most physical spacetimes will not contain any achronal, complete null geodesics. As discussed in [141], this is highly constraining, preventing the formation of closed time-like curves and various other causal pathologies [119, 69].

In fact, this connection between causality and energy conditions is more than a coincidence. The ANEC was proved for conformal field theories with a twist-gap in flat space using the notion of micro-causality, which states that operators commute at space-like separation

$$[O(x), O(x')] = 0 \tag{1.1.4}$$

for $(x - x')^2 > 0$ [75].[1]

In the context of AdS/CFT, micro-causality can be translated into the bulk as a constraint on the bulk geometry. In order for a consistent bulk-boundary dictionary, it must be the case that null curves traveling through the bulk cannot travel faster than curves which stay entirely on the boundary. This condition, which we refer to as the boundary causality condition (BCC) follows from the bulk averaged null curvature condition [66]. In Chapter 3, we will show that the boundary causality condition is one of three related geometric constraints that the bulk must obey in order to have a consistent CFT dual.

In Chapter 2, we will also examine the boundary causality condition in the context of holographic theories lying on fluctuating branes near the asymptotic boundary of AdS. We will find that this is related to the ANEC for the semi-classical theory on the brane as well as a stronger and more local condition

$$\int_\gamma \rho(\lambda) \langle T_{kk}(\lambda) \rangle \, d\lambda \geq -\frac{1}{8\pi G_N} \int_\gamma \frac{(\rho')^2}{\rho} d\lambda \tag{1.1.5}$$

where $\rho(\lambda) \geq 0$ is some smearing function with support only over affine parameter values where γ is achronal.

The ANEC is a non-local condition in that it requires integrating over a complete null geodesic. For many purposes, we require a more local constraint on the null energy. We now turn to reviewing recent developments in this direction.

1.2 Quantum Focussing and the Quantum Null Energy Condition

To uncover a local constraint on energy density, we should return to the most local constraint discussed above: the null energy condition. In a D-dimensional theory of Einstein gravity, the null energy condition is actually equivalent to the statement that null congruences always focus in the presence of matter. This equivalency can easily be seen by examining

[1] The proof given in [75] also applies to QFTs with a UV interacting fixed point.

Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\theta_{(k)}^2 - \sigma_{(k)}^{ab}\sigma_{(k)ab} - R_{kk} \tag{1.2.1}$$

where $k^a = \left(\frac{d}{d\lambda}\right)^a$ and θ_k, σ_k^{ab} are the expansion and shear of the null congruence generated by the null vector field $k^a(y)$. Using Einstein's equations, we can swap null curvature for null energy - $R_{kk} = 8\pi G_N T_{kk}$ - and we see that positivity of T_{kk} ensures negativity of $\frac{d\theta}{d\lambda}$. Conversely, by picking a null congruence with vanishing expansion and shear at some point p, we see that negativity of $\frac{d\theta}{d\lambda}$ implies positivity of T_{kk} at p.

This suggests that finding a quantum analog of the null energy condition is tantamount to finding a quantum generalization of focussing. This led [21] to conjecture that the correct quantum generalization of focussing can be found by upgrading areas to generalized entropies (times the Planck area). The generalized entropy is given by

$$S_{gen} = \frac{A}{4G\hbar} + S_{out} + (\text{higher curvature terms}) \tag{1.2.2}$$

where A is the area of a co-dimension two entangling surface and S_{out} is the von Neumann entropy of the quantum fields to one side of this entangling surface. In general theories of higher curvature gravity, there are extra terms which are given by the Dong entropy [43]. For now, we ignore these terms but return to them in Chapter 5.

By making the replacement, $A \to 4G\hbar S_{gen}$, we arive at a quantum version of (1.2.1), deemed the quantum focussing conjecture or (QFC). This new inequality requires that we track the generalized entropy as the co-dimension two entangling surface is moved up along a null congruence. As a function of this entangling surface profile along the null congruence, which we denote as $X^+(y)$, the QFC states

$$\frac{\delta}{\delta X^+(y')}\frac{1}{\sqrt{\hbar}}\frac{\delta S_{gen}}{\delta X^+(y)} \leq 0 \tag{1.2.3}$$

where y, y' are internal co-ordinates for the entangling surface. As discussed in [21], this inequality holds for $y \neq y'$ by strong sub-additivity of the von Neumann entropy [110]. We will call the contribution to this inequality when $y = y'$ the "diagonal" piece of the QFC. The diagonal piece of the QFC will be proportional to a delta function in $y - y'$. we will denote the coefficient of this delta function by Θ', where Θ is referred to as the quantum expansion. The diagonal QFC can then be written as [21]

$$\Theta'(y) = \theta' + 4G\hbar\,(S''_{out} - S'_{out}\theta) + \text{higher curvature terms} \leq 0 \tag{1.2.4}$$

where θ is the classical expansion of the null congruence at point y. Here we also denote $S'_{out} \equiv \frac{\delta S_{out}}{\delta V(y)}$. S''_{out} is the diagonal contribution to the second functional derivative of S_{out}. Restricting to vanishing expansion surfaces in Einstein gravity, the QFC reduces to the interesting formula

$$\Theta' = -8\pi G\,\langle T_{kk}\rangle + 4G\hbar S''_{out} \leq 0. \tag{1.2.5}$$

Importantly, the factors of G cancel from both sides of this inequality and we land on a statement involving only $\hbar$. We call this inequality the *quantum null energy condition*.

Shortly after conjecturing this inequality, [27] proved it for free, massless scalar field theory. Then, using techniques very similar to those in [75], the authors of [9] proved this inequality for general QFTs with a UV interacting fixed point. This latter method of proof demonstrated that the QNEC is fundamentally a statement of causality, albeit a more subtle version of causality, where normal time evolution is supplanted by *modular* time evolution (i.e. time evolution with respect to the modular Hamiltonian, which we discuss in the following section).

In the context of AdS/CFT, the quantum null energy condition was first proved in CFTs with a holographic dual by making use of a bulk geometric condition called *entanglement wedge nesting* or EWN. This condition says that two nested boundary subregions - $R_2 \subset R_1$ - their corresponding entanglement wedges in the bulk must also be nested in a space-like fashion. EWN should be viewed as a statement about how bulk causality must respect boundary causality in order for a consistent holographic dictionary. In Chapter 3, we will demonstrate that EWN is the strongest of three geometric constraints, one of which is the boundary causality condition mentioned above. We now turn to understanding more broadly the connection between energy and entanglement that the QFC and QNEC suggest.

1.3 Energy and Entanglement

The quantum null energy condition lower bounds the null energy flowing past a point p by the second shape derivative of von Neumann entropy

$$\langle T_{kk}(p)\rangle \geq \frac{\hbar}{2\pi}S'' \tag{1.3.1}$$

and thus connects energy density with local entanglement density or entanglement "curvature." This suggests that if we want to send information via excitations of quantum fields, then we are forced to expend energy.

Such an inequality is just the most local version in a broad class of inequalities connecting energy with entropy. In the context of quantum field theory, the connection has its origins in the foundational result of Bisognano & Wichmann [16]. This result states that for observers confined to one Rindler wedge, the vacuum density matrix for the quantum fields takes the form of a thermal density matrix with *modular Hamiltonian* $H_{\mathcal{R}}$ given by

$$\sigma_{\mathcal{R}} = e^{-2\pi H_{\mathcal{R}}},\ H_{\mathcal{R}} = \int d^{d-2}y \int_0^\infty dx^+\ x^+\, T_{++}(x^- = 0, x^+, y). \tag{1.3.2}$$

This Hamiltonian is easily recognizable as the boost generator in the right Rindler wedge. The relative entropy between some excited state and the vacuum is defined as

$$S(\rho||\sigma) \equiv -\,\mathrm{Tr}[\rho\log\sigma] + \mathrm{Tr}[\rho\log\rho]. \tag{1.3.3}$$

This can be thought of as a measure of distinguishibility between ρ and $\sigma_{\mathcal{R}}$. The factor of $x+$ in the integrand of (1.3.2) comes from the fact that null energy falling across the Rindler horizon at later x^+ is more easily distinguishable from the thermal vacuum noise for a Rindler observer.[2]

Rindler space is defined as the spacetime region to one side of a flat cut. To study the QNEC, it is important to understand the form of the vacuum modular Hamiltonian for more general regions, whose entangling surface might lie along an arbitrary profile on the Rindler horizon. In Chapter 5, we will present an argument for how the QNEC implies a simple formula for the modular Hamiltonian of such regions. We find

$$H_{\mathcal{R}}[X^+(y)] = \int d^{d-2} \int_{X^(y)}^{\infty} (x^+ - X^+(y)) T_{++}(x^- = 0, x^+, y) \tag{1.3.4}$$

where the entangling surface lies at $x^- = 0, x^+ = X^+(y)$.

The fact that the modular Hamiltonian is an integral of a local operator should not be confused with the fact that for $X^+(y) = 0$ it generates a local flow (boosts). When $X^+(y)$ is a non-trivial function of y, the flow will be highly non-local for operators sufficiently far from the null plane. In fact, very little is known about the flow generated by $H_{\mathcal{R}}[X^+(y)]$. This remains an interesting open research area.

The formula in (1.3.4) was proved in [39] for all QFTs using Tomita-Takesaki theory. The important point of this formula for us will be that if we take two second functional derivatives, we are left only with a diagonal (delta function) contribution. Namely,

$$\frac{\delta^2 H_{\mathcal{R}}[X^+]}{\delta X^+(y) \delta X^+(y')} = T_{++}(y) \delta^{d-2}(y - y'). \tag{1.3.5}$$

.

For non-vacuum modular Hamiltonians, we expect a similar formula with other diagonal and non-diagonal contributions which represent the non-locality of the modular Hamiltonian. In particular, we expect a formula of the form

$$\frac{\delta^2 H^{\psi}_{\mathcal{R}}[X^+]}{\delta X^+(y) \delta X^+(y')} = (T_{++} - Q_{\psi})\, \delta^{d-2}(y - y') + (\text{off-diagonal}). \tag{1.3.6}$$

for some global state $|\psi\rangle$ reduced to one side of the cut $X^+(y)$. Note that the operator Q_ψ is a state-dependent operator. The QNEC them implies that $\langle Q_\psi \rangle_\psi \geq 0$. A formula for this expectation value in free scalar field theory was found in [27].

In Chapters 6 and 7, we will present arguments that for QFTs with a UV interacting fixed point, $\langle Q_\psi \rangle_\psi$ is actually zero for every state. This implies that the QNEC is saturated in every state for interacting theories.[3]

[2]We thank Raphael Bousso for emphasizing this point to us.

[3]The technical definition of a interacting CFT is that the spectrum of primary operators has a twist gap above the stress tensor. This means that there should be no other (uncharged) operators beside the stress tensor that saturate the unitarity bound.

These results suggest that Einstein's equations could be understood as a statement about entropic equilibrium. Some progress has been made in this direction [90, 89, 133, 53]. We will examine this idea in more detail in Chapter 6.

1.4 Outline

We now provide a brief outline of this thesis. We begin in Chapter 2 by generalizing the ANEC to weakly curved states of a holographic coupled to gravity. We prove the inequality in (2.1.1), which proves a conjecture of [59]. Furthermore, we show that this inequality is intimately related to an approximate notion of causality for end-of-the-world branes sitting near the asymptotic boundary of AdS.

We then turn to more fine-grained notions of causality in normal AdS/CFT in Chapter 3. We prove logical connections between the QFC, QNEC and EWN as well as several other bulk and boundary statements.

In Chapter 4, we examine the validity of the QNEC for field theories on more general backgrounds. We do this for holographic field theories, dual to theories with higher curvature terms in the low energy effective action.

The remainder of the thesis will be on precise connections between energy and entanglement. In Chapter 5, we prove equation (1.3.4) by assuming the QNEC. We then focus on QNEC saturation, first in the context of holographic field theories in Chapter 6 and then for more general field theories in Chapter 7.

Chapter 2

Upper and Lower Bounds on the Integrated Null Energy in Gravity

2.1 Introduction and Summary

Many recent constraints on the energy density in quantum field theory [26, 96, 76, 52, 9, 106] were originally conjectured as statements in semiclassical gravity. In gravity, these conditions are motivated by the desire to rule out pathologies like closed timeline curves. By taking the $G_N \to 0$ limit, these bounds sometimes turn into non-trivial statements in quantum field theory, which can then be proved directly with field-theoretic techniques.

Once proven in the field theory, one can often perturbatively lift these field-theoretic statements back to semiclassical gravity. For example, the proof of the quantum null energy condition may be used perturbatively for quantum fields on a curved background, thus proving the quantum focusing conjecture, at least in certain states and limits [21].

On the other hand, it is likely that there are additional restrictions on theories of gravity beyond those which come from quantum field theory on a curved background. Indeed, a recent conjecture by Freivogel & Krommydas [59] asserts that for low energy states in a semiclassical theory of quantum gravity, there should be a semilocal bound on the null-null components of the stress tensor of the form[1]

$$\int_{-\infty}^{\infty} du\ \rho(u)\, \langle T_{uu}(u) \rangle \geq -\frac{1}{32\pi G_N} \int_{-\infty}^{\infty} du\ \frac{\rho'(u)^2}{\rho(u)}, \tag{2.1.1}$$

where $\rho(u)$ is an arbitrary, non-negative smearing function, and the integral is over a null geodesic which is achronal on the support of ρ. Freivogel & Krommydas were not able to fix the numerical factor appearing in this bound, but in this note we determine it. Notice that for a compactly-supported ρ, the $G_N \to 0$ limit leaves the resulting field theory energy density unconstrained. This bound also implies the achronal ANEC when applied to an

[1] Outside of this introductory section we will drop explicit expectation values from the notation, but they should be understood.

inextendible achronal null geodesic, but is far more general since generic spacetimes do not possess inextendible achronal null geodescis [69]. This bound is also similar in flavor to the so-called quantum inequalities that have been proposed by Ford & Roman for theories without gravity [56, 57, 58]. In more than two dimensions, such a semilocal bound on the stress tensor is known to be non-existent [55] within field theory, so it is natural that the that the $G_N \to 0$ limit renders (2.1.1) trivial.

In this note we prove the bound in equation (2.1.1) for holographic field theories that have been perturbatively coupled to gravity using the induced gravity framework on a brane [122, 123, 134, 70, 116]. The reason we use induced gravity is that all of the physics, including the low-energy gravitational physics of the brane, is encoded in the AdS dual. In particular, the induced gravity setup fixes the value for Newton's constant, as well as the higher-curvature gravitational couplings on the brane. The consistency of AdS/CFT automatically encodes certain constraints that would be impossible to guarantee if we just coupled the theory to gravity by hand. For instance, it was shown in [116] that in the induced gravity setup the standard holographic entropy formula correctly computes the generalized entropy from the brane point of view, which is a nontrivial check that the induced gravity formalism encodes desirable constraints.

The main assumption in our proof of (2.1.1) is that bulk physics should respect brane causality:

> **Brane Causality Condition**: The intrinsic brane causal structure cannot be violated by transmitting signals through the bulk.

In ordinary AdS/CFT (where the boundary is at infinity and not a brane at finite location), this condition was proved by Gao & Wald [66] for all asymptotically AdS spacetimes satisfying the averaged null curvature condition. However, any assumption about the bulk geometry is less fundamental than the statement of boundary causality, and one should instead use boundary causality as a basic axiom. That strategy was used in [93] to prove the ANEC for the boundary field theory and in [108] to prove the quantum inequalities. Our techniques are similar to those works, and our assumption is the Brane Causality Condition.

One may question whether the Brane Causality Condition is reasonable, even at the classical level. If the brane were an arbitrary hypersurface at finite position then surely the condition would be violated in most situations. However, the brane gravitational equations of motion save us. As we will review below, in low-energy states the brane extrinsic curvature satisfies $K_{uu} \approx 0$, so that null geodesics in the brane geometry are also null geodesics in the bulk geometry. This removes obvious violations of the Brane Causality Condition that would otherwise exist. This also highlights our earlier point about the consistency of induced gravity: coupling another matter sector to the brane metric without using induced gravity will lead to order-one violations of $K_{uu} \approx 0$, and hence of the Brane Causality Condition. In a highly curved or highly quantum regime one may question the validity of the Brane Causality Condition, but in the semiclassical regime we focus on it should be a good assumption. [2]

[2]Our arguments will not even make full use of the Brane Causality Condition. We only require that it

As a second result, we will separately derive an upper bound on the integrated null energy in gravity, namely

$$\int_{-\infty}^{\infty} du\ \rho(u)\,\langle T_{uu}(u)\rangle \leq \frac{d-2}{32\pi G_N}\int_{-\infty}^{\infty} du\ \frac{\rho'(u)^2}{\rho(u)}, \tag{2.1.2}$$

where d is the dimension of the brane theory. Except for the factor of $d-2$, this bound is like a mirror image of (2.1.1). In fact, this bound is much more general (and more trivial). It follows from an analogous upper bound on the integrated null curvature—obtained by multiplying (2.1.2) by $8\pi G_N$ and using Einstein's equation[3]—that is simply a geometrical consequence of achronality. The curvature inequality holds in any spacetime, even when the spacetime is not dynamical. This is in contrast to (2.1.1), which can be violated in an arbitrary spacetime and therefore represents an actual constraint on the states of a consistent theory of gravity.

We can summarize all of these results in the combined statement

$$\frac{d-2}{4}\int_{-\infty}^{\infty} du\frac{\rho'^2}{\rho} \geq \int_{-\infty}^{\infty} du\,\rho R_{uu} \geq -\frac{1}{4}\int_{-\infty}^{\infty} du\frac{\rho'^2}{\rho}, \tag{2.1.3}$$

valid for a null geodesic which is achronal over the support of ρ. Note that this means that, in the event that we have an inextendible achronal null geodesic, the ANCC and ANEC are actually saturated.

The remainder of this note is laid out as follows: in Section 2.2, we will review the induced gravity formalism in the context of AdS/CFT. In Section 2.3, we will discuss the geometric constraint imposed by brane causality. We will then use this constraint to derive (2.1.1). Then in Section 2.4 we will derive (2.1.2), completing our main results. In Section 2.5 we will evaluate (2.1.1) in some recent traversable wormhole constructions which have appreciable negative energy, checking that it is not violated. Finally, in Section 6.6 we will end with a discussion of the results and possible future directions.

2.2 Review of Induced Gravity on the Brane

In this section we review some facts about the induced gravity scenarios that we will use in our computation. The construction was first used in the works of Randall and Sundrum [122, 123], and the relation to AdS/CFT was emphasized in [134, 70]. The extension beyond bulk Einstein gravity can be found in [116]

We are interested in describing the low-energy physics of a large-N field theory coupled to gravity. Because it is only an effective theory, there is an explicit UV cutoff scale. In

be obeyed in the near-brane region of the bulk.

[3] We freely use Einsten's equation in manipulating our inequalities even when the gravitational theory includes higher curvature terms. The assumption is that Einstein's equation is the leading part of the full gravitational equation of motion, and in low-energy states all higher-curvature terms are suppressed and therefore irrelevant for inequalities.

the holographic description, this means that the asymptotically AdS space dual to the field theory has an explicit cutoff surface located at some finite position of the bulk. We will refer to this cutoff surface as the "brane." The brane naturally has a gravitational action induced on it from the bulk gravity theory, and by "induced gravity" we mean that, except for a few simple counterterms that we will describe below, the gravitational action for the brane consists only of the induced action from the bulk.

To aid the discussion we will introduce the coordinate z normal to the brane in such a way that the metric in the vicinity of the brane is

$$ds^2 = \frac{dz^2 + g_{ij}(x,z)dx^i dx^j}{z^2}, \tag{2.2.1}$$

and the brane is located at $z = z_0$. We consider $g_{ij}(z = z_0)$ to be the physical metric of the brane. This is a rescaling of the induced metric by a factor of z_0^2, which is not the standard convention in induced gravity situations but is a convenient choice of units for our purposes. With this choice of metric the cutoff length scale of the effective field theory on the brane is z_0.

Bulk and Boundary Actions

The total action consists of the bulk action, a generalized Gibbons–Hawking–York brane action, and a brane counterterm action:

$$S_{\text{tot}} = S_{\text{bulk}} + S_{\text{GHY}} + S_{\text{ct}}. \tag{2.2.2}$$

Varying $S_{\text{bulk}} + S_{\text{GHY}}$ gives

$$\delta(S_{\text{bulk}} + S_{\text{GHY}}) = \int_{\text{bulk}} (\text{bulk EOM}) + \int_{\text{brane}} \sqrt{g}\ \mathcal{E}^{ij}\delta g_{ij}. \tag{2.2.3}$$

The GHY term is designed so that variation of the action only depends on δg_{ij} and not its derivatives normal to the brane. Then we see that $\mathcal{E}_{ij}$ contributes to the brane gravitational equations of motion. For bulk Einstein gravity, $\mathcal{E}_{ij}$ is proportional to the Brown–York stress tensor, but in higher-derivative bulk gravity it will have additional terms.

The equation $\mathcal{E}_{ij} = 0$ is a higher-derivative gravitational equation of motion from the brane point of view, even when the bulk just has Einstein gravity. We will see below that, for us, it is the null-null component of this equation that matters. In the next section, when we discuss the counterterm action, we will restrict the set of allowed counterterms so that they do not affect the null-null equations of motion. The reason is that the null-null equations of motion are what ensure that the extrinsic curvature of the brane $K_{uu} \approx 0$, which is important for the Brane Causality Condition.

One important consequence of the induced gravity procedure is that the effective Newton constant on the brane is related to the bulk Newton constant by a simple rescaling:

$$G_{\text{brane}} = (d-2)G_{\text{bulk}} z_0^{d-2} + \cdots \tag{2.2.4}$$

Here the $\cdots$ refer to corrections that come from non-Einstein gravity in the bulk, but they will be suppressed by the size of the higher-curvature bulk couplings [116]. We assume that those couplings are small, namely of the order typically generated by bulk quantum effects. Since we are interested in proving inequalities like (2.1.1), only the leading-order parts of our expressions are important, and so terms like this can be dropped.

We would also like to emphasize that the construction of the brane theory is identical to the first few steps of the standard holographic renormalization procedure [73]. In holographic renormalization, one would introduce counterterms that cancel out the purely geometric parts of $\mathcal{E}_{ij}$, and the part that remains is the holographic stress tensor. Here we do not introduce most of those counterterms (the exceptions are described below), and instead interpret those purely geometric parts of $\mathcal{E}_{ij}$ as the geometric terms in the gravitational equations of motion. The upshot is that the ordinary holographic stress tensor still has the same interpretation in the induced gravity scenario as it does in ordinary AdS/CFT: it is the stress tensor of the matter sector of the theory, and it plays the role of the source in the gravitational equations of motion.

Counterterms

Now we discuss the counterterm action, $S_{\rm ct}$. The purpose of the counterterm action is to fine-tune the values of certain mass parameters in the induced theory which would otherwise be at the cutoff scale. This includes the brane cosmological constant, which can be tuned by adding a term to $S_{\rm ct}$ of the form

$$S_{\rm ct} \supset \int_{\rm brane} \sqrt{g}\, \mathcal{T}, \tag{2.2.5}$$

where the constant $\mathcal{T}$ is known as the tension of the brane.

No other purely gravitational counterterms will be added to the brane action. As mentioned in the introduction, the fact that the brane gravity is induced by the bulk gravity is an important constraint that enforces consistency conditions which are not apparent from the effective field theory point of view. A counterterm proportional to the Einstein–Hilbert action, for example, would change the value of the brane Newton constant away from (2.2.4), and thus take us out of induced gravity. From a more practical point of view, we discussed above that the Brane Causality Condition is sensible because $K_{uu} \approx 0$, and that is enforced by the null-null equation of motion determined by $S_{\rm bulk} + S_{\rm GHY}$. To preserve that condition we need that $S_{\rm ct}$ has a trivial variation with respect to the null-null components of the metric. This is true for the cosmological constant counterterm, and in fact is true for any counterterm that only depends on the metric through the volume element $\sqrt{g}$.

When there are low-dimension scalar operators in the field theory, new counterterms are needed to fine-tune their masses and expectation values. These include terms proportional to $\int_{\rm brane} \sqrt{g}\Phi^2$, where Φ is the bulk field dual to the operator, familiar from the theory of holographic renormalization. Like the cosmological constant term, these only depend on the metric through $\sqrt{g}$, and so we can add them freely. We will not say any more about these kinds of terms, as they are not important for the rest of our analysis.

Brane Equations of Motion

Now that we have discussed the action for the induced gravity system, we can calculate the correct gravitational equation of motion. Since all of the terms in S_{ct} are coupled to the metric through $\sqrt{g}$, the result is simple. We find

$$\mathcal{E}_{ij} \propto g_{ij}, \tag{2.2.6}$$

where the proportionality factor could depend on scalar expectation values.

For Einstein gravity in the bulk, this equation sets the extrinsic curvature to be proportional to the metric:

$$K_{ij} \propto g_{ij}, \tag{2.2.7}$$

where

$$K_{ij} = \frac{1}{2z}\partial_z g_{ij} - \frac{1}{z^2} g_{ij}. \tag{2.2.8}$$

Note that the null-null components of the extrinsic curvature would be set to zero according to this equation. For higher-derivative bulk gravity there will be corrections that we comment on below.

When written in terms of brane quantities, the equation of motion takes the form of Einstein's equation plus corrections:

$$R_{ij} = 8\pi G_{\text{brane}} T_{ij} + \cdots . \tag{2.2.9}$$

The higher-curvature terms in $\cdots$ are suppressed by the brane cutoff scale, and so can be consistently dropped in states where the brane curvature scale is well below the brane cutoff scale.

Finally, we quote one additional fact which follows from standard Gauss–Codazzi-like relations on the brane, and that is the following expression for the normal derivative of the extrinsic curvature:

$$z\partial_z K_{ij} = R_{ij} - \mathcal{R}_{ij} - z^2 K K_{ij} + 2z^2 K_{ik} K^k_j, \tag{2.2.10}$$

where R_{ij} and $\mathcal{R}_{ij}$ are the brane and bulk Ricci tensor, respectively.[4] Together with the brane equation of motion, this equation will allow us to prove (2.1.1) in the next section.

[4] Note that we are raising indices in this equation using g_{ij}, not $\gamma_{ij} = g_{ij}/z^2$.

2.3 Lower Bound from Brane Causality

In this section, we derive the bound in (2.1.1) from the Brane Causality Conditon. The technique is very similar to that used to derive the ANEC [93] and quantum inequalities [108] in AdS/CFT, with the main difference being that the brane is at a finite location in the bulk, rather than at infinity, and its intrinsic and extrinsic geometry are determined by equations of motion.

Consider a future-directed achronal null geodesic segment on the brane (defined according to the brane metric), parametrized by affine parameter λ that takes values in the range $\lambda_0 < \lambda < \lambda_1$. We will define the null coordinate u such that $u = \lambda$, and let v be another null coordinate in the neighborhood of the geodesic such that $v = 0$ and $g_{uv} = -1$ along the geodesic itself. We extend these coordinates into the bulk in an arbitrary way, provided that they remain orthogonal to the z coordinate so that (2.2.1) is respected. The Brane Causality Condition states that any future-directed causal curve anchored to the brane—including those which travel through the bulk—beginning at $(u, v) = (\lambda_0, 0)$ must have its other endpoint in the future of our null geodesic segment according to the causal structure of the brane metric.

To derive (2.1.1), we will construct a causal curve which begins at $(u, v) = (\lambda_0, 0)$ on the brane and travels through the bulk before returning to the brane. The restriction that the curve is causal means that

$$\left(\frac{dZ}{d\lambda}\right)^2 + g_{ij}(X, Z)\frac{dX^i}{d\lambda}\frac{dX^j}{d\lambda} \leq 0, \tag{2.3.1}$$

where $X^i(\lambda)$ and $Z(\lambda)$ are the coordinates of the bulk curve.

To get the strictest bound, we will try to construct a bulk curve which moves as quickly as possible while remaining causal (i.e., gets infinitesimally close to being null in the bulk). Thus, we choose the bulk curve to follow a trajectory very close to the geodesic segment on the brane:

$$z = Z(\lambda) = z_0 + \epsilon\sqrt{\rho(\lambda)}, \tag{2.3.2}$$

$$u = U(\lambda) = \lambda, \tag{2.3.3}$$

$$v = \epsilon^2 V(\lambda). \tag{2.3.4}$$

The function ρ is non-negative, smooth, and satisfies $\rho(\lambda_0) = \rho(\lambda_1) = 0$, but is otherwise arbitrary. Here ϵ is a small length scale, and we should say how small it is relative to the other scales in the problem. Recall that the cutoff scale for the brane theory is z_0, and let us denote the characteristic curvature scale on the brane in the state we consider by ℓ. Then we want our parameters to be such that

$$z_0 \ll \epsilon \ll \ell. \tag{2.3.5}$$

The idea here is that our bulk curve is not probing the deep UV of the theory, where quantum gravity effects may become large, but is still microscopic compared to the curvatures scales of the state we are in. The fact that $\ell \gg z_0$ is part of the semiclassical assumption.

Expanding (2.3.1) in ϵ out to $O(\epsilon^2)$, we find

$$\epsilon\sqrt{\rho}\partial_z g_{uu} + \epsilon^2 \left(\frac{1}{4}\frac{\rho'^2}{\rho} + \frac{1}{2}\rho\partial_z^2 g_{uu} - 2V'\right) \leq 0. \tag{2.3.6}$$

All metric factors are being evaluated at $z = z_0$ along the null geodesic segment. Note that in order for this expansion to make sense we have implicitly assumed that $z_0\rho'/\rho << \epsilon\rho'/\rho << 1$. As a consequence, this restricts the bulk curves from rapidly increasing or decreasing on the scale of the brane cutoff.

Consider the $O(\epsilon)$ term. If the bulk theory were pure Einstein gravity, then from (2.2.7) and (2.2.8) we would have $\partial_z g_{uu} = 0$ on the brane. This would be violated by a small amount in higher-curvature bulk theories. Even in that case, we know from the Fefferman-Graham expansion of the metric that, generally, $\partial_z g_{uu} \propto z_0$ [88, 130]. Thus the $O(\epsilon)$ term is negligible for multiple reasons compared to the $O(\epsilon^2)$ term, and so we can consistently drop it from the inequality.

For the $O(\epsilon^2)$ term, the main problem is evaluating $\partial_z^2 g_{uu}$ on the brane. This is easily accomplished using (2.2.10), along with the brane equations of motion. In the case of bulk Einstein gravity, from (2.2.7) we find that

$$\frac{1}{2}\partial_z^2 g_{uu} = R_{uu} - \mathcal{R}_{uu}. \tag{2.3.7}$$

For non-Einstein gravity in the bulk, there will be small corrections to this equation proportional to the bulk curvature couplings. But since those couplings are small, all of those correction terms can be dropped while preserving the inequality.

We find that (2.3.1) reduces to

$$\frac{1}{4}\frac{\rho'^2}{\rho} + \rho\left(R_{uu} - \mathcal{R}_{uu}\right) - 2V' \leq 0. \tag{2.3.8}$$

We can satisfy this condition by choosing

$$V(\lambda) = \frac{1}{2}\int_{\lambda_0}^{\lambda} \rho\left(R_{uu} - \mathcal{R}_{uu}\right) d\tilde{\lambda} + \frac{1+\delta}{8}\int_{\lambda_0}^{\lambda} \frac{\rho'^2}{\rho} d\tilde{\lambda} \tag{2.3.9}$$

Here $\delta > 0$ is a regulator that we will eventually take to zero. Thus the total change in the v coordinate over the entire trajectory is

$$\Delta v = \epsilon^2 \left(\frac{1}{2}\int_{\lambda_0}^{\lambda_1} \rho\left(R_{uu} - \mathcal{R}_{uu}\right) d\tilde{\lambda} + \frac{1+\delta}{8}\int_{\lambda_0}^{\lambda_1} \frac{\rho'^2}{\rho} d\tilde{\lambda}\right). \tag{2.3.10}$$

Now we impose the Brane Causality Condition, which demands that $\Delta v \geq 0$. This must be true for any δ, so in the limit $\delta \to 0$ we find the inequality

$$\int_{\lambda_0}^{\lambda_1} \rho\left(R_{uu} - \mathcal{R}_{uu}\right) d\tilde{\lambda} \geq -\frac{1}{4}\int_{\lambda_0}^{\lambda_1} \frac{\rho'^2}{\rho} d\tilde{\lambda}. \tag{2.3.11}$$

We are free to formally let $\lambda_0 \to -\infty$ and $\lambda_1 \to +\infty$ as long as the geodesic is achronal on the support of ρ.

Now we will argue that $\mathcal{R}_{uu}$ should be dropped from the inequality, which will complete the proof. From the bulk equations of motion, $\mathcal{R}_{uu} \approx 8\pi G_{\text{bulk}} T^{\text{bulk}}_{uu}$. When written in terms of expectation values of operators in the brane field theory the slowest possible falloff at small z_0 is $T^{\text{bulk}}_{uu} \propto z_0^{2\Delta}$ with $2\Delta > d-2$ by the unitarity bound. On the other hand, $R_{uu} \approx 8\pi G_{\text{brane}} T_{uu}$ and $G_{\text{brane}} \sim z_0^{d-2}$ from (2.2.4). Thus at small z_0 the $\mathcal{R}_{uu}$ term is negligible, and we recover (2.1.1).

2.4 Upper Bound From Achronality

In this section, we note that achronality actually also implies an upper bound on the null curvature. This bound will be purely geometric and apply equally well to dynamical and non-dynamical backgrounds, though in theories of gravity we can turn it into the bound (2.1.2) on the null energy density.

The setup is the same as before, where we have a future-directed achronal null geodesic segment with affine parameter λ such that $\lambda_0 < \lambda < \lambda_1$. Choose some smooth function $\rho(\lambda)$ such that $\rho(\lambda_0) = \rho(\lambda_1) = 0$. We will assume that λ_0 and λ_1 are both finite at first, and we will allow them to go to infinity later as part of a limiting procedure. Then we can perform the Weyl transformation

$$g_{ij} \to \tilde{g}_{ij} = \rho^{-1} g_{ij} \tag{2.4.1}$$

in a neighborhood of the segment (after choosing some suitable extension of the affine parameter to that neighborhood). Since causal structure is preserved by Weyl transformations, in the new spacetime our segment is actually an inextendible achronal null geodesic. Note that λ no longer affinely-parameterizes the geodesic, but we can pick a new affine parameter $\tilde{\lambda}$ defined by the generator $\tilde{k}^i = (\partial_{\tilde{\lambda}})^i = \rho k^i$, where $k^i = (\partial_\lambda)^i$ is the generator in the original spacetime. The endpoints of the geodesic are at $\tilde{\lambda} = \pm\infty$, which confirms that the geodesic is inextendible.

A key fact is that the conformal transformation properties of the Ricci curvature imply that

$$\int_{\lambda_0}^{\lambda_1} d\lambda \left(\rho R_{ij} k^i k^j - \frac{d-2}{4} \frac{\rho'^2}{\rho} \right) = \int_{-\infty}^{\infty} d\tilde{\lambda}\ \tilde{R}_{ij} \tilde{k}^i \tilde{k}^j. \tag{2.4.2}$$

Thus to prove (2.1.2) we only have to show that the integrated null curvature on the right-hand-side is negative. Since we are assuming λ_0 and λ_1 are finite—and that the curvature in the original spacetime does not have singularities—we see from the expression on the left-hand-side that the integrated null curvature in the new spacetime is bounded.

Since our geodesic is inextendible and achronal in the new spacetime, it must be that a null congruence starting at $\tilde{\lambda} = -\infty$ with vanishing expansion (and twist) does not encounter a caustic at any point along the geodesic, meaning that the expansion $\tilde{\theta}$ remains finite as a function of $\tilde{\lambda}$. Integrating Raychaudhuri's equation gives

$$\tilde{\theta}(+\infty) = \int_{-\infty}^{\infty} d\tilde{\lambda} \left(-\frac{\tilde{\theta}^2}{d-2} - \tilde{\sigma}^2 - \tilde{R}_{ij}\tilde{k}^i\tilde{k}^j \right). \tag{2.4.3}$$

If the integrated null curvature is positive, then $\tilde{\theta}(+\infty)$ is negative. But then the integral of $\tilde{\theta}^2$ diverges and we learn that actually $\tilde{\theta}(+\infty)$ itself is divergent. By making the same argument at large-but-finite $\tilde{\lambda}$, we can also rule out the possibility that $\tilde{\theta}$ oscillates between positive and negative values as it diverges. We will now show that $\tilde{\theta}$ cannot diverge at infinity, which proves the result.

Under the assumption that $\tilde{\theta}$ diverges at infinity, consider dividing Raychaudhuri's equation by $\tilde{\theta}^2$ first and then integrating from some $\tilde{\lambda}_0$ to $\tilde{\lambda}_1$, with $\tilde{\lambda}_0$ chosen large enough so that $\tilde{\theta}$ does not vanish for any $\tilde{\lambda} > \tilde{\lambda}_0$. We find

$$\frac{1}{\tilde{\theta}_0} - \frac{1}{\tilde{\theta}_1} + \int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1} d\tilde{\lambda}\, \frac{\tilde{R}_{ij}\tilde{k}^i\tilde{k}^j}{\tilde{\theta}^2} = -\frac{\tilde{\lambda}_1 - \tilde{\lambda}_0}{d-2} - \int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1} d\tilde{\lambda}\, \frac{\tilde{\sigma}^2}{\tilde{\theta}^2}. \tag{2.4.4}$$

Given the finiteness of the integrated null curvature, we see that the left-hand-side of this equation goes to a constant as $\tilde{\lambda}_1 \to \infty$ while the right-hand-side diverges. Thus we have proved the inconsistency of $\tilde{\theta}$ diverging at infinity, and the desired result follows.

2.5 Applications

We now discuss possible applications of this bound to semiclassical gravity. In the regime of weak gravity, we might worry that the bound is trivial because $1/G_N$ is large compared to the size of the stress tensor. However, we can make up for this if the geodesic is long enough. Clearly in the case of an infinite geodesic the bound (2.1.1) implies the achronal ANEC, which is not a trivial statement. For finite but long geodesics we can get relatively strong lower bounds by choosing ρ to slowly ramp up from zero to one, say by choosing $\rho = (\lambda - \lambda_0)^2/(\Delta\lambda)^2$ for some interval $\lambda_0 < \lambda < \lambda_0 + \Delta\lambda$, before transitioning to $\rho = 1$. Then the integral of ρ'^2/ρ is of order $1/\Delta\lambda$. Thus if $\Delta\lambda \sim 1/G_N$ we can get $O(G_N^0)$ lower bounds on the integrated null energy, assuming that most of the null energy flux is in the part of the geodesic where $\rho = 1$.

In the remainder of this section we will apply the above strategy to two recent constructions of traversable wormhole solutions, which make critical use of negative energy. We will see how the achronality condition prevents each from violating (2.1.1).

Gao–Jafferis–Wall Wormhole

In [65] a wormhole in the bulk is made traversable by coupling two holographic CFTs in the thermofield double state. The coupling breaks achronality of the black hole horizon, thereby allowing negative averaged null energy along the horizon without violating the achronal

ANEC. However, (2.1.1) still applies, and we can see what consequences it has. This is a case where the stress tensor is perturbative and $O(N^0)$ in large-N counting, while the lower bound is $O(N^2)$. One might hope that applying the strategy above to reduce the magnitude of the lower bound would help here, but it does not: one can check that in situations where the geodesic becomes long enough to appreciably decrease the magnitude of the lower bound, the magnitude of the integrated energy flux decreases by an even larger factor.[5] Thus the bound never becomes tight for this construction.

Maldacena–Milekhin–Popov Wormhole

In [111] the authors constructed a traversable wormhole in four-dimensional asymptotically flat space threaded by magnetic flux and supported by the negative Casimir energy of a fermion field. The wormhole interior is given by an approximate $AdS_2 \times S_2$ metric,

$$ds^2 \approx r_e^2 \left(-(1+\xi^2)\frac{dt^2}{\ell^2} + \frac{d\xi^2}{1+\xi^2} + d\Omega_2^2 \right), \tag{2.5.1}$$

where r_e parameterizes the size of the wormhole and ℓ is such that the t coordinate smoothly maps onto the Minkowski t coordinate outside the wormhole. This metric is only a good description for $|\xi| \lesssim \xi_c \sim \ell/r_e \gg 1$, where it opens up into the asymptotically flat ambient space.

We can use ξ as the affine parameter of a null geodesic that passes through the wormhole, and we need to integrate the null Casimir energy along the geodesic. From solving Einstein's equations, one learns that there is a relationship between the energy density and the parameter ξ_c. The end result is $\ell^2 T_{tt} = (1+\xi^2)^2 T_{\xi\xi} \sim -1/G_N\xi_c$, which means that the integrated null energy is

$$\int d\xi \, \left(T_{\xi\xi} + \ell^2 \frac{T_{tt}}{(1+\xi^2)^2} \right) \sim -\frac{1}{G_N \xi_c} \,, \tag{2.5.2}$$

with most of the contribution coming from the region $\xi \lesssim 1$.

Naively, one would consider a geodesic which went through the entire wormhole, $-\xi_c < \xi < \xi_c$, and by appropriately choosing $\rho(\xi)$ one could make $\int \rho'^2/\rho \sim 1/\xi_c$. In that case we would parametrically saturate (2.1.1), and it would be up to the order-one coefficients to determine if the bound were in danger of being violated. However, this is too fast and we first need to properly account for the achronality condition.

In the ambient flat space, the two ends of the wormhole are a proper distance d apart, which means it takes a time d to send a signal from one to the other. Sending a signal through the wormhole would take a time

$$\int_{-\xi_c}^{\xi_c} \frac{\ell d\xi}{1+\xi^2} \approx \pi\ell, \tag{2.5.3}$$

[5]We thank Don Marolf for discussions on this point.

which one expects to be greater than d so that the wormhole respects the ambient causality. Define $y = \pi\ell/d$. In the solutions of [111] the minimal value of y was approximately 2.35, and $y = 1$ means that ambient causality is being saturated.

If $y > 1$ then it is faster to travel through the ambient space than it is through the wormhole, and so the null geodesic which passes through the entire wormhole from end to end is not achronal. In order to maintain achronality, we need to restrict the null geodesic segment to lie within the range $|\xi| < \xi_1$ where

$$\arctan \xi_1 = \frac{\pi}{4}\left(1 + \frac{1}{y}\right) - \frac{1}{2\xi_c}, \tag{2.5.4}$$

in the approximation that $\xi_c \gg 1$. We see that when $y = 1$ we have $\xi_1 \sim \xi_c$ and (2.1.1) would be parametrically saturated, if not violated. However, if y is appreciably larger than 1, as it is in [111], then $\xi_1 \sim 1$ and we are far from saturating (2.1.1). Thus it seems that (2.1.1) is intimately connected with causality in the ambient space.

2.6 Discussion and Future Directions

The obvious next goal would be to prove (2.1.1) without using induced gravity. Our method of proof involved an extension of bulk-boundary causality to the brane at $z = z_0$. This suggests that the bound (2.1.1) is to be related to some notion of causality in the gravitational theory. In [3], it was shown that the analogous condition in ordinary AdS/CFT was implied by the principle of entanglement wedge nesting. Furthermore, in [9] it was shown that entanglement wedge nesting can be re-cast as a statement of causality under modular time evolution. It would be interesting to understand if (2.1.1) is related to some notion of modular causality in effective gravitational theories. An investigation along these lines would also have to confront the fact that the naive generalization of entanglement wedge nesting to the brane case is almost always violated.

Recently, the bound of [46], which provided a bulk geometric condition for good bulk-boundary causality to hold in asymptotically AdS spacetimes, was given a CFT understanding by looking at the Regge limit of boundary OPEs [1]. It seems reasonable that one could use similar techniques to prove the bulk version of (2.1.1).

Finally, it would be surprising if this bound were logically separate from the Quantum Focusing Conjecture [21]. Unlike the QFC and related results, the entropy is conspicuously absent from (2.1.1). The lack of any $\hbar$ factors suggest that (2.1.1) is more classical than those other bounds,[6] but we leave an exploration of a possible relationship to future work.

[6] We thank Raphael Bousso for emphasizing this point.

Chapter 3

Geometric Constraints from Subregion Duality

3.1 Introduction

AdS/CFT implies constraints on quantum gravity from properties of quantum field theory. For example, field theory causality requires that null geodesics through bulk are delayed relative to those on the boundary. Such constraints on the bulk geometry can often be understood as coming from energy conditions on the bulk fields. In this case, bulk null geodesics will always be delayed as long as there is no negative null energy flux [66].

In this paper, we examine two constraints on the bulk geometry that are required by the consistency of the AdS/CFT duality. The starting point is the idea of subregion duality, which is the idea that the state of the boundary field theory reduced to a subregion A is itself dual to a subregion of the bulk. The relevant bulk region is called the entanglement wedge, $\mathcal{E}(A)$, and consists of all points spacelike related to the extremal surface on the side towards A [42, 81]. The validity of subregion duality was argued to follow from the Ryu-Takayanagi-FLM formula in [44, 72], and consistency of subregion duality immediately implies two more bulk conditions beyond the BCC.

The first condition, which we call Entanglement Wedge Nesting (EWN), is that if a region A is contained in a region B on the boundary (or more generally, if the domain of dependence of A is contained in the domain of dependence of B), then $\mathcal{E}(A)$ must be contained in $\mathcal{E}(B)$.

The second condition is that the set of bulk points in $I^-(D(A)) \cap I^+(D(A))$, called the causal wedge $\mathcal{C}(A)$, is completely contained in the entanglement wedge $\mathcal{E}(A)$. We call this $\mathcal{C} \subseteq \mathcal{E}$.

In section 3.2 we will spell out the definitions of EWN and $\mathcal{C} \subseteq \mathcal{E}$ in more detail, as well as describe their relations with subregion duality. Roughly speaking, EWN encodes the fact that subregion duality should respect inclusion of boundary regions. $\mathcal{C} \subseteq \mathcal{E}$ is the statement that the bulk region dual to a given boundary region should at least contain all those bulk points from which messages can be both received from and sent to the boundary region.

Even though EWN, $\mathcal{C} \subseteq \mathcal{E}$, and the BCC are all required for consistency of AdS/CFT, part of our goal is to investigate their relationships to each other as bulk statements independent of AdS/CFT. As such, we will demonstrate that EWN implies $\mathcal{C} \subseteq \mathcal{E}$, and $\mathcal{C} \subseteq \mathcal{E}$ implies the BCC. Thus EWN is in a sense the strongest statement of the three.

Though this marks the first time that the logical relationships between EWN, $\mathcal{C} \subseteq \mathcal{E}$, and the BCC have been been independently investigated, all three of these conditions are known in the literature and have been proven from more fundamental assumptions in the bulk. In the classical limit, a common assumption about the bulk physics is the Null Energy Condition (NEC). However, the NEC is known to be violated in quantum field theory. Recently, the Quantum Focusing Conjecture (QFC), which ties together geometry and entropy, was put forward as the ultimate quasi-local "energy condition" for the bulk, replacing the NEC away from the classical limit [21].

The QFC is the strongest reasonable quasi-local assumption that one can make about the bulk dynamics, and indeed we will show below that it can be used to prove EWN. There are other, weaker, restrictions on the bulk dynamics which follow from the QFC. The Generalized Second Law (GSL) of horizon thermodynamics is a consequence of the QFC. In [47], it was shown that the GSL implies what we have called $\mathcal{C} \subseteq \mathcal{E}$. Thus the QFC, the GSL, EWN, and $\mathcal{C} \subseteq \mathcal{E}$ form a square of implications. The QFC is the strongest of the four, implying the truth of the three others, while the EWCC is the weakest. This pattern continues in a way summarized by Figure 3.1, which we will now explain.

In the first column of Figure 3.1, we have the QFC, the GSL, and the Achronal Averaged Null Energy Condition (AANEC). As we have explained, the QFC is the strongest of these three and the AANEC is the weakest. In the second column we have EWN, $\mathcal{C} \subseteq \mathcal{E}$, and the BCC. In addition to the relationships mentioned above, it was shown in [66] that the ANEC implies the BCC, which we extend to prove the BCC from the AANEC.

The third column of Figure 3.1 contains "boundary" versions of the first column: the QNEC, the QHANEC, and the boundary AANEC[1]. These are field theory statements which can be viewed as nongravitational limits of the corresponding statements in the first column. The QNEC is the strongest, implying the QHANEC, which in turn implies the AANEC. All three of these statements can be formulated in non-holographic theories, and all three are expected to be true generally. (The AANEC was recently proven in [52] as a consequence of monotonicity of relative entropy and in [76] as a consequence of causality.)

In the case of a holographic theory, it was shown in [96] that EWN in the bulk implies the QNEC for the boundary theory to leading order in $G\hbar \sim 1/N$. We demonstrate that this relationship continues to hold under quantum corrections. Moreover, in [93] the BCC in the bulk was shown to imply the boundary AANEC. Here we will complete the pattern of implications by showing that $\mathcal{C} \subseteq \mathcal{E}$ implies the boundary QHANEC.

The remainder of this paper is organized as follows. In Section 3.2 we will carefully define all of the statements we set out to prove, as well as establish notation. Then in Sections

[1]For simplicity we are assuming throughout that the boundary theory is formulated in Minkowski space. There would be additional subtleties with all three of these statements if the boudnary were curved.

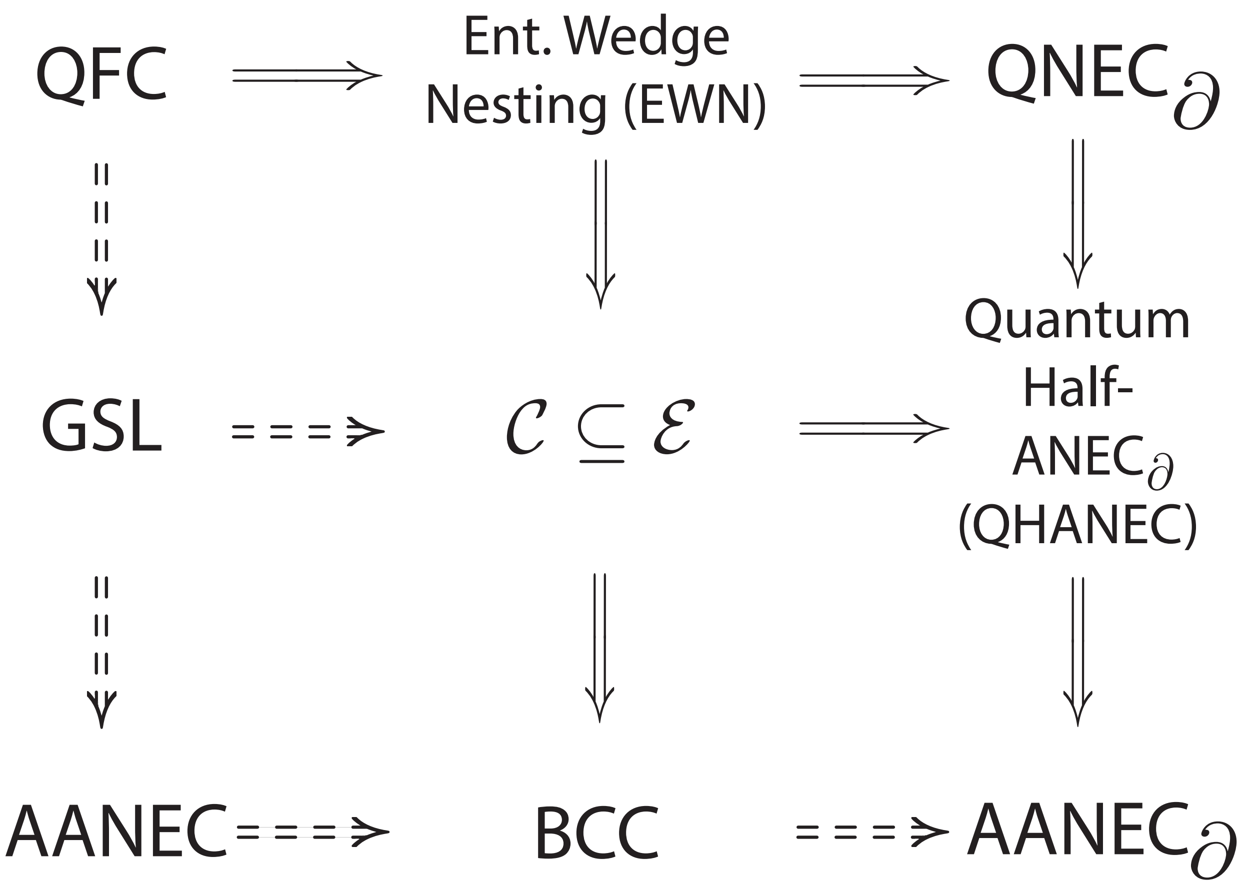

Figure 3.1: The logical relationships between the constraints discussed in this paper. The left column contains semi-classical quantum gravity statements in the bulk. The middle column is composed of constraints on bulk geometry. In the right column is quantum field theory constraints on the boundary CFT. All implications are true to all orders in $G\hbar \sim 1/N$. We have used dashed implication signs for those that were proven to all orders before this paper.

3.3 and 3.4 we will prove all of the implications present in Figure 3.1. Several of these implications are already established in the literature, but for completeness we will briefly review the relevant arguments. We conclude with a discussion in Section 6.6.

3.2 Glossary

Semiclassical Expansion Quantum gravity is a tricky subject. We work in a semiclassical regime, where the dynamical fields can be expanded perturbatively in $G\hbar$ about a classical background [138]. For example, the metric has the form

$$g_{ab} = g^0_{ab} + g^{1/2}_{ab} + g^1_{ab} + O((G\hbar)^{3/2}) \ , \tag{3.2.1}$$

where the superscripts denote powers of $G\hbar$. In the semi-classical limit — defined as $G\hbar \to 0$ — the validity of the various inequalities we consider will be dominated by their leading non-vanishing terms. We assume that the classical $O((G\hbar)^0)$ part of the metric satisfies the NEC, without assuming anything about the quantum corrections.

We primarily consider the case where the bulk theory can be approximated as Einstein gravity with minimally coupled matter fields. In the semiclassical regime, bulk loops will generate Planck-suppressed higher derivative corrections to the gravitational theory and the gravitational entropy [2]. We will comment on the effects of these corrections throughout.

Geometrical Constraints

There are a number of known properties of the AdS bulk causal structure and extremal surfaces. At the classical level (i.e. at leading order in $G\hbar \sim 1/N$), the Null Energy Condition is the standard assumption made about the bulk which ensures that these properties are true [140]. However, some of these are so fundamental to subregion duality that it is sensible to demand them and to ask what constraints in the bulk might ensure that these properties hold even under quantum corrections. That is one key focus of this paper.

In this section, we review three necessary geometrical constraints. In addition to defining each of them and stating their logical relationships (see Figure 3.1), we explain how each is crucial to subregion duality.

Boundary Causality Condition (BCC)

A standard notion of causality in asymptotically-AdS spacetimes is the condition that *the bulk cannot be used for superluminal communication relative to the causal structure of the boundary.* More precisely, any causal bulk curve emanating from a boundary point p and

[2] Such corrections are also necessary for the generalized entropy to be finite. See Appendix A of [21] for details and references. Other terms can be generated from, for example, stringy effects, but these will be suppressed by ℓ_s. For simplicity, we will not separately track the ℓ_s expansion. This should be valid as long as the string scale is not much different from the Planck scale.

arriving back on the boundary must do so to the future of p as determined by the boundary causal structure.

This condition, termed "BCC" in [46], is known to follow from the averaged null curvature condition (ANCC) [66]. Engelhardt and Fischetti have derived an equivalent formulation in terms of an integral inequality for the metric in the context of linearized perturbations to the vacuum [46].

A concrete reason to require the BCC in AdS/CFT is so that microcausality in the CFT is respected. If the BCC were violated, a bulk excitation could propagate between two spacelike-separated points on the boundary leading to nonvanishing commutators of local fields at those points. In Sec. 3.4 we will show that BCC is implied by $\mathcal{C} \subseteq \mathcal{E}$. Thus BCC is the weakest notion of causality in holography that we consider.

$\mathcal{C} \subseteq \mathcal{E}$

Consider the domain of dependence $D(A)$ of a boundary region A. Let us define the causal wedge of a boundary region A to be $I^-(D(A)) \cap I^+(D(A))$.

By the Ryu-Takayanagi-FLM formula, the entropy of the quantum state restricted to A is given by the area of the extremal area bulk surface homologous to A plus the bulk entropy in the region between that surface and the boundary. This formula was shown to hold at $O((1/N)^0)$ in the large-N expansion. In [47], Engelhardt and Wall proposed that an all-orders modification of this formula is to replace the extremal area surface with the Quantum Extremal Surface (QES), which is defined as the surface which extremizes the sum of the surface area and the entropy in the region between the surface and A. Though the Engelhardt-Wall prescription remains unproven, we will assume that it is the correct all-orders prescription for computing the boundary entropy of A. We denote the QES of A as $e(A)$.

The entanglement wedge $\mathcal{E}(A)$ is the bulk region spacelike-related to $e(A)$ on the A side of the surface. This is the bulk region believed to be dual to A in subregion duality. It was argued in [44] that this is the case using the formalism of quantum error correction.

$\mathcal{C} \subseteq \mathcal{E}$ is the property that *the entanglement wedge $\mathcal{E}(A)$ associated to a boundary region A completely contains the causal wedge associated to A.* An equivalent definition of $\mathcal{C} \subseteq \mathcal{E}$ states that $e(A) \cap (I^+(D(A)) \cup I^-(D(A)) = \emptyset$. In our proofs below we will use this latter characterization.

Subregion duality requires $\mathcal{C} \subseteq \mathcal{E}$ because the bulk region dual to a boundary region A should at least include all of the points that can both send and receive causal signals to and from $D(A)$. Moreover, if $\mathcal{C} \subseteq \mathcal{E}$ were false then it would be possible to use local unitary operators in $D(A)$ to send a bulk signal to $e(A)$ and thus change the entropy associated to the region. That is, of course, not acceptable.

This condition has been discussed at the classical level in [81, 140]. In the semiclassical regime, Engelhardt and Wall [47] have shown that it follows from the generalized second law (GSL) of causal horizons. We will show in Sec. 3.4 that $\mathcal{C} \subseteq \mathcal{E}$ is also implied by Entanglement Wedge Nesting.

Entanglement Wedge Nesting (EWN)

The strongest of the geometrical constraints we consider is EWN. In the framework of subregion duality, EWN is the property that a strictly larger boundary region should be dual to a strictly larger bulk region. More precisely, *for any two boundary regions A and B with domain of dependence $D(A)$ and $D(B)$ such that $D(A) \subset D(B)$, we have $\mathcal{E}(A) \subset \mathcal{E}(B)$.*

This property was identified as important for subregion duality and entanglement wedge reconstruction in [42, 140], and was proven by Wall at leading order in G assuming the null curvature condition [140]. We we will show in Sec. 3.4 that the Quantum Focusing Condition (QFC) [21] implies EWN in the semiclassical regime assuming the generalization of HRT advocated in [47].

Constraints on Semiclassical Quantum Gravity

Reasonable theories of matter are often assumed to satisfy various energy conditions. The least restrictive of the classical energy conditions is the null energy condition (NEC), which states that

$$T_{kk} \equiv T_{ab}\, k^a k^b \geq 0 \ , \tag{3.2.2}$$

for all null vectors k^a. This condition is sufficient to prove many results in classical gravity. In particular, many proofs hinge on the classical focussing theorem [136], which follows from the NEC and ensures that light-rays are focussed whenever they encounter matter or gravitational radiation:

$$\theta' \equiv \frac{d}{d\lambda}\theta \leq 0 \ , \tag{3.2.3}$$

where θ is the expansion of a null hypersurface and λ is an affine parameter.

Quantum fields are known to violate the NEC, and therefore are not guaranteed to focus light-rays. It is desirable to understand what (if any) restrictions on sensible theories exist in quantum gravity, and which of the theorems which rule out pathological phenomenon in the classical regime have quantum generalizations. In the context of AdS/CFT, the NEC guarantees that the bulk dual is consistent with boundary microcausality [66] and holographic entanglement entropy [140, 32, 80, 81], among many other things.

In this subsection, we outline three statements in semiclassical quantum gravity which have been used to prove interesting results when the NEC fails. They are presented in order of increasing strength. We will find in sections 3.3 and 3.4 that each of them has a unique role to play in the proper functioning of the bulk-boundary duality.

Achronal Averaged Null Energy Condition

The achronal averaged null energy condition (AANEC) [137] states that

$$\int T_{kk}\, d\lambda \geq 0 \ , \tag{3.2.4}$$

where the integral is along a complete achronal null curve (often called a “null line”). Local negative energy density is tolerated as long as it is accompanied by enough positive energy density elsewhere. The *achronal* qualifier is essential for the AANEC to hold in curved spacetimes. For example, the Casimir effect as well as quantum fields on a Schwarzschild background can both violate the ANEC [95, 135] for chronal null geodesics. An interesting recent example of violation of the ANEC for chronal geodesics in the context of AdS/CFT was studied in [65].

The AANEC is fundamentally a statement about quantum field theory formulated in curved backgrounds containing complete achronal null geodesics. It has been proven for QFTs in flat space from monotonicity of relative entropy [52], as well as causality [76]. Roughly speaking, the AANEC ensures that when the backreaction of the quantum fields is included it will focus null geodesics and lead to time delay. This will be made more precise in Sec. 3.4 when we discuss a proof of the boundary causality condition (BCC) from the AANEC.

Generalized Second Law

The generalized second law (GSL) of horizon thermodynamics states that the generalized entropy (defined below) of a causal horizon cannot decrease in time.

Let Σ denote a Cauchy surface and let σ denote some (possibly non-compact) codimension-2 surface dividing Σ into two distinct regions. We can compute the von Neumann entropy of the quantum fields on the region outside of σ, which we will denote S_{out}[3]. The generalized entropy of this region is defined to be

$$S_{\text{gen}} = S_{grav} + S_{\text{out}} \tag{3.2.5}$$

where S_{grav} is the geometrical/gravitational entropy which depends on the theory of gravity. For Einstein gravity, it is the familiar Bekenstein-Hawking entropy. There will also be Planck-scale suppressed corrections[4], denoted Q, such that it has the general form

$$S_{grav} = \frac{A}{4G\hbar} + Q \tag{3.2.6}$$

There is mounting evidence that the generalized entropy is finite and well-defined in perturbative quantum gravity, even though the split between matter and gravitational entropy depends on renormalization scale. See the appendix of [21] for details and references.

[3]The choice of “outside” is arbitrary. In a globally pure state both sides will have the same entropy, so it will not matter which is the “outside.” In a mixed state the entropies on the two sides will not be the same, and thus there will be two generalized entropies associated to the same surface. The GSL, and all other properties of generalized entropy, should apply equally well to both.

[4]There will also be stringy corrections suppressed by α'. As long as we are away from the stringy regime, these corrections will be suppressed in a way that is similar to the Planck-suppressed ones, and so we will not separately track them.

The *quantum expansion* Θ can be defined (as a generalization of the classical expansion θ) as the functional derivative per unit area of the generalized entropy along a null congruence [21]:

$$\Theta[\sigma(y); y] \equiv \frac{4G\hbar}{\sqrt{h}} \frac{\delta S_{\text{gen}}}{\delta \sigma(y)} \tag{3.2.7}$$

$$= \theta + \frac{4G\hbar}{\sqrt{h}} \frac{\delta Q}{\delta \sigma(y)} + \frac{4G\hbar}{\sqrt{h}} \frac{\delta S_{\text{out}}}{\delta \sigma(y)} \tag{3.2.8}$$

where $\sqrt{h}$ denotes the determinant of the induced metric on σ, which is parametrized by y. These functional derivatives denote the infinitesimal change in a quantity under deformations of the surface at coordinate location y along the chosen null congruence. To lighten the notation, we will often omit the argument of Θ.

A future (past) causal horizon is the boundary of the past (future) of any future-infinite (past-infinite) causal curve [91]. For example, in an asymptotically AdS spacetime any collection of points on the conformal boundary defines a future and past causal horizon in the bulk. The generalized second law (GSL) is the statement that the quantum expansion is always nonnegative towards the future on any future causal horizon

$$\Theta \geq 0 \ , \tag{3.2.9}$$

with an analogous statement for a past causal horizon.

In the semiclassical $G\hbar \to 0$ limit, Eq. (3.2.7) reduces to the classical expansion θ if it is nonzero, and the GSL becomes the Hawking area theorem [77]. The area theorem follows from the NEC.

Assuming the validity of the GSL allows one to prove a number of important results in semiclassical quantum gravity [142, 47]. In particular, Wall has shown that it implies the AANEC [141], as we will review in Section 3.3, and $\mathcal{C} \subseteq \mathcal{E}$ [47], reviewed in Section 3.4 (see Fig. 3.1).

Quantum Focussing Conjecture

The Quantum Focussing Conjecture (QFC) was conjectured in [21] as a quantum generalization of the classical focussing theorem, which unifies the Bousso Bound and the GSL. The QFC states that the functional derivative of the quantum expansion along a null congruence is nowhere increasing:

$$\frac{\delta \Theta[\sigma(y_1); y_1]}{\delta \sigma(y_2)} \leq 0 \ . \tag{3.2.10}$$

In this equation, y_1 and y_2 are arbitrary. When $y_1 \neq y_2$, only the S_{out} part contributes, and the QFC follows from strong subadditivity of entropy [21]. For notational convenience, we will often denote the "local" part of the QFC, where $y_1 = y_2$, as[5]

$$\Theta'[\sigma(y); y] \leq 0. \tag{3.2.11}$$

Note that while the GSL is a statement only about causal horizons, the QFC is conjectured to hold on any cut of any null hypersurface.

If true, the QFC has several non-trivial consequences which can be teased apart by applying it to different null surfaces [21, 18, 47]. In Sec. 3.4 we will see that EWN can be added to this list.

Quantum Null Energy Condition

When applied to a locally stationary null congruence, the QFC leads to the Quantum Null Energy Condition (QNEC) [21, 96]. Applying the Raychaudhuri equation and Eqs. (3.2.5), (3.2.7) to the statement of the QFC (5.2.4), we find

$$0 \geq \Theta' = -\frac{\theta^2}{D-2} - \sigma^2 - 8\pi G\, T_{kk} + \frac{4G\hbar}{\sqrt{\hbar}}\,(S''_{\text{out}} - S'_{\text{out}}\theta) \tag{3.2.12}$$

where S''_{out} is the local functional derivative of the matter entropy to one side of the cut. If we consider a locally stationary null hypersurface satisfying $\theta^2 = \sigma^2 = 0$ in a small neighborhood, this inequality reduces to the statement of the *Quantum Null Energy Condition* (QNEC) [21]:

$$T_{kk} \geq \frac{\hbar}{2\pi}\, S''_{\text{out}} \tag{3.2.13}$$

It is important to notice that the gravitational coupling G has dropped out of this equation. The QNEC is a statement purely in quantum field theory which can be proven or disproven using QFT techniques. It has been proven for both free fields [26] and holographic field theories at leading order in $G\hbar$ [96].[6] In Section 3.4 of this paper, we generalize this proof to all orders in $G\hbar$. These proofs strongly suggest that the QNEC is a true property of quantum field theory in general.[7]

In the classical $\hbar \to 0$ limit, the QNEC becomes the NEC.

[5]Strictly speaking, we should factor out a delta function $\delta(y_1 - y_2)$ when discussing the local part of the QFC [26, 96]. Since the details of this definition are not important for us, we will omit this in our notation.

[6]There is also evidence [63] that the QNEC holds in holographic theories where the entropy is taken to be the casual holographic information [85], instead of the von Neumann entropy.

[7]The free-field proof of [26] was for arbitrary cuts of Killing horizons. The holographic proof of [96] (generalized in this paper) showed the QNEC for a locally stationary ($\theta = \sigma = 0$) portion of *any* Cauchy-splitting null hypersurface in flat space.

Quantum Half-Averaged Null Energy Condition

The *quantum half-averaged energy condition* is an inequality on the integrated stress tensor, and the first null derivative of the entropy on one side of any locally-stationary Cauchy-splitting surface subject to a causality condition(described below):

$$\int_{\lambda}^{\infty} T_{kk}\, d\lambda' \geq -\frac{\hbar}{2\pi} S'(\lambda), \tag{3.2.14}$$

where k^a generates a null congruence with vanishing expansion and shear in a neighborhood of the geodesic and λ is the affine paramter along the geodesic. The geodesic thus must be of infinite extent and have $R_{ab}k^a k^b = C_{abcd}k^a k^c = 0$ everywhere along it. The aforementioned causality condition is that the Cauchy-splitting surfaces used to define $S(\lambda)$ should not be timelike-related to the half of the null geodesic T_{kk} is integrated over. Equivalently, $S(\lambda)$ should be well-defined for all λ from the starting point of integration all the way to $\lambda = \infty$.

The causality condition and the stipulation that the null geodesic in (3.2.14) be contained in a locally stationary congruence ensures that the QHANEC follows immediately from integrating the QNEC (Eq. (3.2.13)) from infinity (as long as the entropy isn't evolving at infinite affine parameter, i.e., $S'(\infty) = 0$). Because the causality condition is a restriction on the global shape of the surface, there will be situation where the QNEC holds but we cannot integrate to arrive at a QHANEC.

The QHANEC appears to have a very close relationship to monotonicity of relative entropy. Suppose that the modular Hamiltonian of the portion of a null plane above an arbitrary cut $\sigma(y)$ is given by

$$K[\sigma(y)] = \int d^{d-2}y \int_{\sigma(y)} d\lambda\ (\lambda - \sigma(y))\, T_{kk} \tag{3.2.15}$$

Then (3.4.25) becomes monotonicity of relative entropy. As of yet, there is no known general proof in the literature of (3.2.15), though for free theories it follows from the enhanced symmetries of null surface quantization [138]. Eq. (3.2.15) can be also be derived for holographic field theories [98]. It has also been shown that linearized backreaction from quantum fields obeying the QHANEC will lead to a spacetime satisfying the GSL [138].[8]

In Sec. 3.4, we will find that $C \subseteq E$ implies the QHANEC on the boundary.

[8] It has been shown [28] that holographic theories also obey the QHANEC when the causal holographic information [85] is used, instead of the von Neumann entropy. This implies a second law for the causal holographic information in holographic theories

3.3 Relationships Between Entropy and Energy Inequalities

GSL implies AANEC

Here we expand on a proof given by Wall in [141]. The proof given in that reference only works for perturbations to a classical spacetime where the null energy condition holds. Here we will prove that the GSL implies the AANEC for AAdS spacetimes.

The reasoning is as follows: for any spacetime where the AANEC is saturated, we will show that the GSL implies the AANEC for perturbations to that spacetime. This shows that the GSL gives the AANEC on any connected region of phase space that includes a state where the AANEC holds.[9] Here by "connected" we mean connected within the semiclassical approximation.

We start by proving that for any achronal null geodesic γ, there exists a congruence containing γ for which $\theta^2 = 0$ and $\sigma^2 = 0$ along γ.

Since γ is a null achronal geodesic, it must be contained in some past causal horizon, $\mathcal{H}^-$. Since $\mathcal{H}^-$ is a causal horizon, $\theta(\lambda) \to 0$ as $\lambda \to -\infty$. By integrating Raychaudhuri's equation, we know that[10]

$$\theta(\lambda) = -\int_{-\infty}^{\lambda} \left(\frac{\theta^2}{D-2} + \sigma^2\right) d\lambda' - 8\pi G \int_{-\infty}^{\lambda} T_{kk}\, d\lambda'. \tag{3.3.1}$$

In the future null direction on $\mathcal{H}^-$, γ will not leave the horizon, because it is achronal. Therefore it cannot reach any caustics before $\lambda \to \infty$. Thus, either $\lim_{\lambda\to\infty} |\theta(\lambda)| < \infty$ or $\theta(\lambda) \to -\infty$ as $\lambda \to \infty$. Because we are assuming that $\int T_{kk}\, d\lambda$ is zero, then T_{kk} must fall off faster than $1/\lambda$ as $\lambda \to \pm\infty$. Thus, if $\theta(\lambda)$ does not also die off accordingly, it will blow up to $\theta \to -\infty$ in finite time. Thus, $\theta(\lambda)$ goes to zero at $\lambda \to \infty$.

Then by taking $\lambda \to \infty$ in (3.3.1) and using that the AANEC is saturated on this null geodesic in this background, we find that both θ and σ must be zero for all values of λ. This fact is all that is needed to continue the proof of the AANEC from the GSL. The remainder follows without modification from [141].

QFC implies GSL In a manner exactly analogous to the proof of the area theorem from classical focusing, the QFC can be applied to a causal horizon to derive the GSL. Consider integrating Eq. 5.2.4 from future infinity along a generator of a past causal horizon:

$$\int d^{d-2}y\, \sqrt{h} \int_{\lambda}^{\infty} d\lambda'\, \Theta'[\sigma(y,\lambda), y] \leq 0 \tag{3.3.2}$$

[9] There may be separate, connected regions of phase space where the AANEC never holds. This proof does not rule out that scenario.

[10] Loop (higher derivative) corrections to the equations of motion will be subleading. Here we also do not worry about negativity of the operator σ^2. Any negative fluctuations come from graviton contributions, which we absorb into the definition of the stress tensor.

Along a future causal horizon, $\theta \to 0$ as $\lambda \to \infty$, and it is reasonable to expect the matter entropy S_{out} to stop evolving as well. Thus $\Theta \to 0$ as $\lambda \to \infty$, and the integrated QFC then trivially becomes

$$\Theta[\lambda(y); y] \geq 0 \tag{3.3.3}$$

which is the GSL.

QHANEC implies AANEC In flat space, all achronal null geodesics lie on a null plane. Applying the QHANEC to cuts of this null plane taking $\lambda \to -\infty$ produces the AANEC, Eq. (3.2.4).

3.4 Relationships Between Entropy and Energy Inequalities and Geometric Constraints

EWN implies $\mathcal{C} \subseteq \mathcal{E}$ implies the BCC

EWN implies $\mathcal{C} \subseteq \mathcal{E}$

We prove the contrapositive. Consider an arbitrary region A on the boundary. $\mathcal{C} \subseteq \mathcal{E}$ is violated if and only if there is at least one $p \in e(A)$ such that $p \in \mathcal{T}(A)$. This implies that there is a timelike curve connecting $e(A)$ to $D(A)$, and hence there exists an open ball of points $\mathcal{O}$ in $D(A)$ that is timelike related to $e(A)$ (see Figure 3.4). Consider a new boundary region $B \subset \mathcal{O}$. It follows that $e(B)$ contains points that are also timelike related to $e(A)$. Therefore Entanglement Wedge Nesting is violated.

$\mathcal{C} \subseteq \mathcal{E}$ implies the BCC

We prove the contrapositive. Without loss of generality, take the boundary of AdS to have topology $S^{d-1} \times \mathbb{R}$. Then the null geodesics originating from an arbitrary point p_- on the boundary of AdS will reconverge at the point p_+. If the BCC is violated, then there exists some null geodesic from p_- through the bulk that arrives at a point q on the boundary *to the past of* p_+. Hence there exists an open neighborhood of points $\mathcal{O}$ around p_+ such $\mathcal{O} \subset I^+(q)$. Choose a boundary region A such that the boundary of A is in $\mathcal{O}$ and $p_- \in D(A)$. Then $e(A)$ will contain at least some points that are timelike related to p_- (see 3.4), and therefore $\mathcal{C} \subseteq \mathcal{E}$ is violated.

Semiclassical Quantum Gravity Constraints Imply Geometric Constraints

Quantum Focussing implies Entanglement Wedge Nesting

This proof will follow closely that laid out in [47]. Consider a boundary region A with associated boundary domain of dependence $D(A)$. As above, we denote the quantum extremal

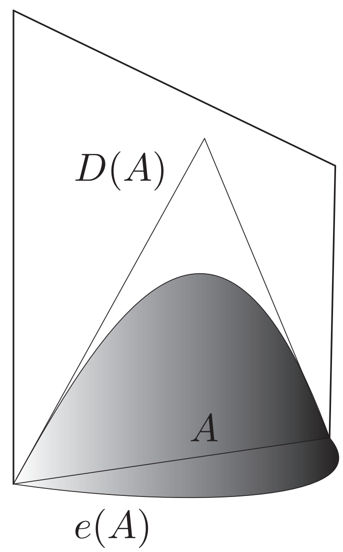

Figure 3.2: The causal relationship between $e(A)$ and $D(A)$ is pictured in an example spacetime that violates $\mathcal{C} \subseteq \mathcal{E}$. The boundary of A's entanglement wedge is shaded. Notably, in $\mathcal{C} \subseteq \mathcal{E}$ violating spacetimes, there is necessarily a portion of $D(A)$ that is timelike related to $e(A)$. Extremal surfaces of boundary regions from this portion of $D(A)$ are necessarily timelike related to $e(A)$, which violates EWN.

surface anchored to ∂A as $e(A)$. For any other boundary region, B, such that $D(B) \subset D(A)$, we will show that $\mathcal{E}(B) \subset \mathcal{E}(A)$, assuming the QFC.

The QFC implies that the null congruence generating the boundary of $I^{\pm}(e(A))$ satisfies $\dot{\Theta} \leq 0$. Combined with $\Theta = 0$ at $e(A)$ (from the definiton of quantum extremal surface), this implies that every point on the boundary of $\mathcal{E}(A)$ satisfies $\Theta \leq 0$. Therefore the boundary of $\mathcal{E}(A)$ is a quantum extremal barrier as defined in [47], and no quantum extremal surfaces can intersect it. This forbids any extremal surfaces $e(B)$ from containing points outside of $\mathcal{E}(A)$ for $D(B) \subset D(A)$. Therefore $e(B) \subset \mathcal{E}(A)$, and by extension $\mathcal{E}(B) \subset \mathcal{E}(A)$.

Generalized Second Law implies $\mathcal{C} \subseteq \mathcal{E}$

This proof can be found in [47], but we elaborate on it here to illustrate similarities between this proof and the proof that QFC implies EWN.

Wall's Lemma We remind the reader of a fact proved as Theorem 1 in [142]. Let two boundary anchored co-dimension two, space-like surfaces M and N contains the point $\{p\} \in M \cap N$ such that they are also tangent at p. Both surfaces are Cauchy-splitting in the bulk AdS. Suppose that $M \subseteq Ext(N)$. In the classical regime, Wall shows that there exists some point x in a neighborhood of p where either

$$\theta_N(x) > \theta_M(x) \tag{3.4.1}$$

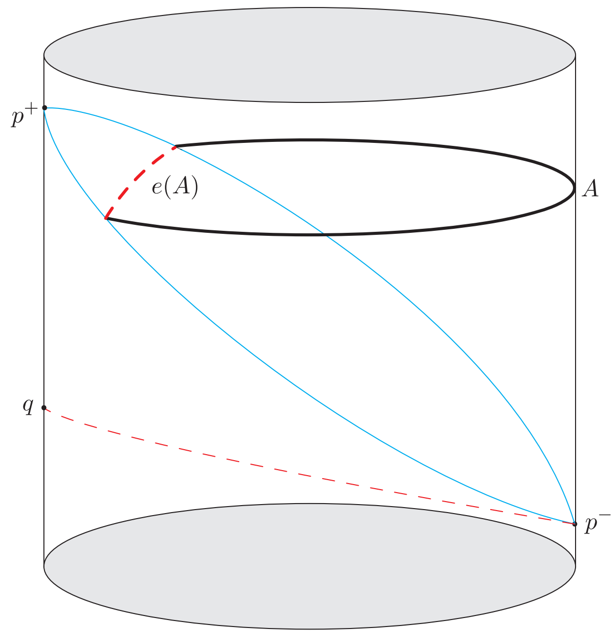

Figure 3.3: A violation of $\mathcal{C} \subseteq \mathcal{E}$ is depicted as a consequence of the failure of the BCC. A null geodesic connects p_- and q through the bulk (thin red dashed line). The boundary of $I^+(p_-)$ is depicted on the boundary (blue lines). The extremal surface $e(A)$ is timelike related to q, which contradicts $\mathcal{C} \subseteq \mathcal{E}$.

or the two surfaces actually agree everywhere in the neighborhood. These expansions are associated to the exterior facing, future null normal direction.

In the semi-classical regime, this result can be improved to bound the quantum expansions

$$\Theta_1(x) > \Theta_2(x) \tag{3.4.2}$$

where x is some point in a neighborhood of p. The proof of this quantum result requires the use of strong sub-additivity, and works even when bulk loops generate higher derivative corrections to the generalized entropy [142].

We now proceed by contradiction. Suppose that the causal wedge lies at least partly outside the entanglement wedge. In this discussion, by the "causal wedge," we mean the intersection of the past of $I^-(\partial D(A))$ with the Cauchy surface on which $e(A)$ lies. Then by continuity, we can shrink the boundary region associated to the causal wedge. At some point, the causal wedge must shrink inside the entanglement wedge boundary. The configuration that results is reproduced in Figure 3.4. The causal wedge will be inside of $e(A)$ and tangent at some point p.

At this point, by the above lemma, the generalized expansions should obey

$$\Theta_e(x) > \Theta_c(x) \tag{3.4.3}$$

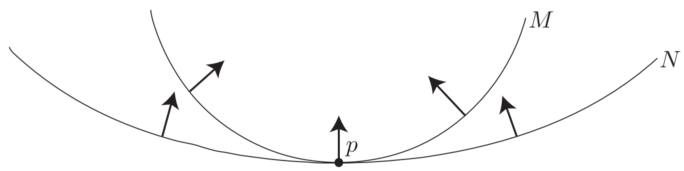

Figure 3.4: The surface M and N are shown touching at a point p. In this case, $\theta_M < \theta_N$. The arrows illustrate the projection of the null orthogonal vectors onto the Cauchy surface.

for x in some neighborhood of p. Assuming genericity of the state, the two surfaces cannot agree in this neighborhood. The Wall-Engelhardt prescription tells us that the entanglement wedge boundary should be given by the quantum extremal surface [47] and so

$$\Theta_e(x) = 0 > \Theta_c(x) \tag{3.4.4}$$

Thus, the GSL is violated at some point along this causal surface, which draws the contradiction.

AANEC implies Boundary Causality Condition

The Gao-Wald proof of the BCC assumes that all complete null geodesics through the bulk contain a pair of conjugate points [66] . The standard focusing theorem ensures that this follows from the NEC and the null generic condition (discussed below) [136]. Here, we sketch a slight modification of the proof which instead assumes the achronal averaged null energy condition (AANEC).

We prove that the AANEC implies BCC by contradiction. Let the spacetime satisfy the null generic condition [136], so that each null geodesic encounters at least some matter or gravitational radiation.[11] Assume that the BCC is violated, so that there exists at least one complete achronal null geodesic γ through the bulk connecting two boundary points. The AANEC, along with the generic condition, requires that $T_{kk} \geq 0$ somewhere along γ. However, in 3.3 we showed that along such achronal null geodesics, $\theta = \sigma = 0$ everywhere. This implies $\dot{\theta} = -T_{kk}$, which from the generic condition implies that $\dot{\theta} \neq 0$ somewhere, which is a contradiction.

Geometric Constraints Imply Field Theory Constraints

Entanglement Wedge Nesting implies the Boundary QNEC

At leading order in $G\hbar \sim 1/N$, this proof is the central result of [96]. There the boundary entropy was assumed to be given by the RT formula without the bulk entropy corrections. We give a proof here of how the $1/N$ corrections can be incorporated naturally. We will now show, in a manner exactly analogous to that laid out in [96], that EWN implies the boundary

[11]Mathematically, each complete null geodesic should contain a point where $k^a k^b k_{[c} R_{d]ab[e} k_{f]} \neq 0$.

QNEC. In what follows, we will notice that in order to recover the boundary QNEC, we must use the *quantum extremal surface*, not just the RT surface with FLM corrections [47].

The essential ida here will be to take bulk quantities "to the boundary." This will become clear momentarily.

The quantum extremal surface prescription, as first introduced in [47], says that the entropy of a region, A, in the boundary CFT is given by first finding the minimal generalized entropy region homologous to A. Then the entropy formula then says

$$S_{\mathrm{b}dry} = S_{\mathrm{gen},min} = \frac{A_{\mathrm{Q}ES}}{4G\hbar} + S_{\mathrm{bul}k} \tag{3.4.5}$$

Entanglement Wedge Nesting then becomes a statement about how the quantum extremal surface moves under deformations to the boundary region. In particular, for null variations of the boundary region, EWN states that the bulk QES moves in a spacelike (or null) fashion.

To state this more precisely, we can set up a null orthogonal basis about the QES. Let k^μ be the inward-facing, future null orthogonal vector along the quantum extremal surface. Let ℓ^μ be its past facing partner with $\ell \cdot k = 1$. Following the prescrition in [96], we denote the locally orthogonal deviation vector of the quantum extremal surface by s^μ. This vector can be expanded in the local null basis as

$$s = \alpha k + \beta \ell \tag{3.4.6}$$

The statement of entanglement wedge nesting then just becomes the statement that $\beta \geq 0$.

In order to find how β relates to the boundary QNEC we would like to find its relation to the entropy. We start by examining the expansion of the extremal surface solution in Fefferman-Graham coordinates. Note that the quantum extremal surface obeys an equation of motion including the bulk entropy term as a source

$$K_\mu = -\frac{4G\hbar}{\sqrt{H}} \frac{\delta S_{\mathrm{bul}k}}{\delta X^\mu} \tag{3.4.7}$$

Here, $K^\mu = \theta_k \ell^\mu + \theta_\ell k^\mu$ is the extrinsic curvature of the QES. As discussed in [96], solutions to (3.4.7) without the bulk source take the form

$$\bar{X}^i_{\mathrm{HR}T}(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + ... + \frac{z^d}{d}(V^i(y^a) + W^i(y^a)\log z) + o(z^d) \tag{3.4.8}$$

We now claim that the terms lower order than z^d are unaffected by the presence of the source. More precisely

$$\bar{X}^i_{\mathrm{Q}ES}(y^a, z) = X^i(y^a) + \frac{1}{2(d-2)} z^2 K^i(y^a) + ... + \frac{z^d}{d}(V^i_{\mathrm{Q}ES} + W^i(y^a)\log z) + o(z^d) \tag{3.4.9}$$

This expansion can be found by examining the leading order pieces of the extremal surface equation. For the quantum extremal surface equation in (3.4.7), we find the same equation as in [96] but with a source:

$$z^{d-1}\partial_z\left(z^{1-d} f \sqrt{\bar{h}} \bar{h}^{zz} \partial_z \bar{X}^i\right) + \partial_a\left(\sqrt{\bar{h}_{ab}} \bar{h}^{ab} f \partial_b \bar{X}^i\right) = -z^{d-1} 4G\hbar f \frac{\delta S_{bulk}}{\delta \bar{X}^j} g^{ji} \tag{3.4.10}$$

Here we are paremeterizing the near-boundary AdS metric in Fefferman-Graham coordinates by

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \left[f(z)\eta_{ij} + \frac{16\pi G_N}{dL^{d-1}}z^d t_{ij}\right]dx^i dx^j + o(z^d)\right). \tag{3.4.11}$$

The function $f(z)$ encodes the possibility of relevant deformations in the field theory which take us away from pure AdS.

One can then plug in the expansion in (3.4.9) into (3.4.10) to see that the lower order than z^d terms remain unaffected by the presence of the bulk entropy source as long as $\delta S_{\text{bulk}}/\delta X^i$ remains finite at $z = 0$. We will encounter a similar condition on derivatives of the bulk entropy below. We discuss its plausibility at the end of this section.

For null perturbations to locally stationary surfaces on the boundary, one can show using (3.4.9) that the leading order piece of β in the Fefferman-Graham expansion arrives at order z^{d-2}. In fact [96],

$$\beta \propto z^{d-2}\left(T_{kk} + \frac{L^{d-1}}{8\pi G_N}k_i\partial_\lambda V^i_{\text{QES}}\right). \tag{3.4.12}$$

We will now show that V^i_{QES} is proportional to the variation in S_{gen} at all orders in $1/N$, as long as one uses the quantum extremal surface and assumes mild conditions on derivatives of the bulk entropy. The key will be to leverage the fact that S_{gen} is extremized on the QES. Thus, its variation will come from pure boundary terms. At leading order in z, we will identify these boundary terms with the vector V_{QES}.

We start by varying the generalized entropy with respect to a boundary perturbation

$$\delta S_{\text{gen}} = \int_{\text{QES}} \frac{\delta S_{gen}}{\delta \bar{X}^i}\delta\bar{X}^i dz d^{d-2}y - \int_{z=\epsilon}\left(\frac{\partial S_{\text{gen}}}{\partial(\partial_z \bar{X}^i)} + ...\right)\delta\bar{X}^i d^{d-2}y \tag{3.4.13}$$

where the boundary term comes from integrating by parts when deriving the Euler-Lagrange equations for the functional $S_{\text{gen}}[\bar{X}]$. The ellipsis denotes terms involving derivatives of S_{gen} with respect to higher derivatives of the embedding functions $(\partial S_{\text{gen}}/\partial(\partial^2 X), \ldots)$ These boundary terms will include two types terms: one involving derivatives of the surface area and one involving derivatives of the bulk entropy.

The area term was already calculated in [96]. There it was found that

$$\frac{\partial A}{\partial(\partial_z \bar{X}^i)} = -\frac{L^{d-1}}{z^{d-1}}\int d^{d-2}y\sqrt{h}\frac{g_{ij}\partial_z\bar{X}^i}{\sqrt{1 + g_{lm}\partial_z\bar{X}^l\partial_z\bar{X}^m}}\delta\bar{X}^j|_{z=\epsilon} \tag{3.4.14}$$

One can use (3.4.9) to expand this equation in powers ϵ, and then contract with the vector null vector k on the boundary in order to isolate the variation with respect to null deformations. For boundary surfaces which a locally stationary some point y, one finds that all terms lower order than z^d vanish at y. In fact, it was shown in [96] that the right hand side of (3.4.14), after contracting with k^i, is just $k^i V_i$ at first non-vanishing order. As for the bulk entropy terms in (3.4.13), in order for them to not affect the boundary QNEC, we need to make the assumption that these derivatives all vanish as $z \to 0$. We have used similar assumptions about the vanishing of entropy variations at infinity throughout this paper. The final result is that

$$k^i V_i^{QES} = -\frac{1}{L^{d-1}\sqrt{h}} k^i \frac{\delta S_{\text{gen}}}{\delta X^i} \tag{3.4.15}$$

The quantum extremal surface prescription says that the boundary field theory entropy is just equal to the generalized entropy of the QES [47]. Setting $S_{\text{gen}} = S_{\text{bdry}}$ in (3.4.15) and combining that with (3.4.12) shows that the condition $\beta \geq 0$ is equivalent to the QNEC. Since the EWN guarantess that $\beta \geq 0$, the proof is complete.

We briefly comment about the assumptions used to derive (3.4.15). The bulk entropy should - for generic states - not depend on the precise form of the region near the boundary. The intuition is clear in the thermodynamic limit where bulk entropy is extensive. As long as we assume strong enough fall-off conditions on bulk matter, the entropy will have to go to zero as $z \to 0$.

Note here the importance of using the quantum extremal surface and not just the HRT surface with bulk entropy corrections added in by hand. Had we naively continued to use the HRT, we would have discovered a correction to the boundary QNEC from the bulk entropy. In other words, if one wants to preserve the logical connections put forth in Figure 3.1, the use of quantum extremal surfaces is necessary.

We discuss loop corrections in the form of higher derivative corrections to the gravitational action at the end of this section.

$C \subseteq E$ implies the QHANEC

Much like the proof above, we examine the statement of $\mathcal{C} \subseteq \mathcal{E}$ near the boundary. This proof will also hold to all orders in $1/N$, again assuming proper fall conditions on derivatives of the bulk entropy.

The basic idea will be to realize that general states in AdS/CFT can be treated as perturbations to the vacuum in the limit of small z. Again, we will consider the general case where the boundary field theory includes relevant deformations. Then the near the boundary the metric can be written

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \left[f(z)\eta_{ij} + \frac{16\pi G_N}{dL^{d-1}} z^d t_{ij}\right] dx^i dx^j + o(z^d)\right), \tag{3.4.16}$$

where $f(z)$ encodes the effects of the relevant deformations. In this proof we take the viewpoint that the order z^d piece of this expansion is a perturbation on top of the vacuum. In other words

$$g_{ab} = g_{ab}^{vac} + \delta g_{ab}. \tag{3.4.17}$$

Of course, this statement is highly coordinate dependent. In the following calculations, we treat the metric as a field on top of fixed coordinates. We will have to verify the gauge-independence of the final result, and do so below.

We begin the proof by taking the boundary region of interest to be the half space, $A = \{X^i | x \geq 0,\ t = 0\}$. The boundary of this space clearly lies at $x = t = 0$.[12]

In vacuum, we need to verify that the quantum extremal surface $e(A)$ lies on the past causal horizon in the bulk. For the classical surface, the solution can be calculated directly. For the quantum extremal surface, the structure of Lorentz symmetries on the vacuum guarantees this fact as well. An arbitrary, wiggly cut of a null plane can be deformed back to a flat cut by action with an infinite boost. Such a transformation preserves the vacuum, and so by demanding continuity of the the QES under this boost, we find that the extremal surface must have been on the Poincare horizon. Had the QES partly left the Poincare horizon, then it would have been taken off to infinity by the boost.[13]

Since the extremal surface lies on the null plane, one can construct an orthogonal null coordinate system around the QES in the vacuum. We denote the null orthogonal vectors by k and ℓ where $k^z = 0 = \ell^z$ and $k^x = k^t = 1$ so that $k \cdot \ell = 1$. Then the statement of $\mathcal{C} \subseteq \mathcal{E}$ becomes [14]

$$k \cdot (\eta - \bar{X}_{SD}) \geq 0 \tag{3.4.18}$$

Here we use $\eta, \bar{X}_{SD}$ to denote the perturbation of the causal horizon and QES surface from their vacuum position, respectively. The notation of $\bar{X}_{SD}$ is used to denote the state dependent piece of the embedding functions for the extremal surface. Over-bars will denote bulk embedding functions of the QES surface and X^a will denote boundary coordinates. The set up is illustrated in 3.4.

Just as in the previous section, for a locally stationary surface such as the wiggly cut of a null plane, one can write the embedding coordinates of the QES, $\bar{X}$, as an expansion in z [96].

$$\bar{X}^i(y, z) = X^i(y) + \frac{1}{d} V^i(y) z^d + o(z^d) \tag{3.4.19}$$

[12] What follows would also hold for regions whose boundary is an arbitary cut of a null plane. In null coordinates, that looks like $\partial A = \{(u \geq U_0(y), v = 0)\}$. All we need to hold is that the extremal surface lies on the Poincare horizon in the vacuum. The same argument given in the body for flat cuts of a null plane should still hold in the general case.

[13] It is also worth noting that EWN together with $\mathcal{C} \subseteq \mathcal{E}$ can also be used to construct an argument. Suppose we start with a flat cut of a null plane, for which the QES is also a flat cut of a null plane in the vacuum. We can then deform this cut on the boundary to an arbitrary, wiggly cut of the null plane in its future. In the bulk, EWN states that the QES would have to move in a space-like or null fashion, but if it moves in a space-like way, then $\mathcal{C} \subseteq \mathcal{E}$ is violated.

[14] The issue of gauge invariance for this proof should not be overlooked. On their own, each term in (3.4.18) is not gauge invariant under a general diffeomorphism. The sum of the two, on the other hand, does not transform under coordinate change:

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)}$$

Plugging this into the formula for $k \cdot \eta$ shows that $\delta(k \cdot \eta) = -(k \cdot \xi)$, which is precisely the same as the change in position of the extremal surface $\delta(k \cdot \bar{X}_{SD}) = -(k \cdot \xi)$.

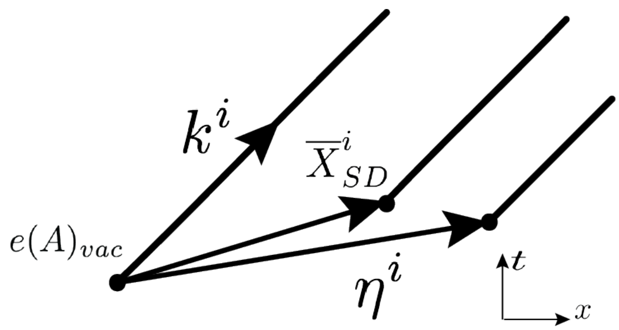

Figure 3.5: This picture shows the various vectors defined in the proof. It depicts a cross-section of the extremal surface at constant z. $e(A)_{vac}$ denotes the extremal surface in the vacuum. For flat cuts of a null plane on the boundary, they agree. For wiggly cuts, they will differ by some multiple of k^i.

where V^i is some local "velocity" function that denotes the rate at which the entangling surface diverges from its boundary position. In vacuum, $V^i \propto k^i$, and so for non-vacuum states $k \cdot \bar{X}_{SD} = \frac{1}{d} V \cdot k z^d + o(z^d)$.

Equation (3.4.15) tells us that $\bar{X}_{SD}$ is proportional to boundary variations of the CFT entorpy. Thus, equation (3.4.19) together with (3.4.15) tells us the simple result that

$$k \cdot \overline{X}_{SD} = -\frac{4G_N}{dL^{d-1}\sqrt{h}} S'_{CFT} z^{d-2} \tag{3.4.20}$$

where S_k is the variation of the entropy under null deformations of the boundary region.

Now we explore the η deformation. This discussion follows much of the formalism found in [46]. At a specific value of (z, y), the null generator of the causal surface will have a different tangent vector, related to k by

$$k' = k + \delta k = k + k^a \nabla_a \eta \tag{3.4.21}$$

In the perturbed metric, k' must be null to leading order in $\eta = \mathcal{O}(z^d)$. Imposing this condition we find that

$$k^b \nabla_b (\eta \cdot k) = -\frac{1}{2} \delta g_{ab} k^a k^b \tag{3.4.22}$$

This equation can be integrated back along the original null geodesic, with the boundary condition imposed that $\eta(\infty) = 0$. Thus, we find the simple relation that

$$(k \cdot \eta)(\lambda) = \frac{1}{2} \int_\lambda^\infty \delta g_{kk} \, d\lambda \tag{3.4.23}$$

The holographic dictionary gives us a nice relation between this integral and boundary quantities. Namely, to leading order in z, the expression above can be recast in terms of the CFT stress tensor

$$k \cdot \eta = \frac{1}{2} \int_\lambda^\infty \frac{16\pi G_N}{dL^{d-3}} z^{d-2} T_{kk} \, d\lambda \tag{3.4.24}$$

Plugging all of this back in to (3.4.18), we finally arrive at the basic inequality

$$\int_{\lambda}^{\infty} T_{kk}\, d\lambda + \frac{\hbar}{2\pi\sqrt{\hbar}} S'_{CFT} \geq 0 \tag{3.4.25}$$

Note that all the factors of G_N have dropped out and we have obtained a purely field theoretic QHANEC.

Loop corrections Here we will briefly comment on why bulk loop corrections affect the argument. Quantum effects do not just require that we add S_{out} to A; higher derivative terms suppressed by the Planck-scale will be generated in the gravitational action which will modify the gravitational entropy functional. With Planck-scale suppressed higher derivative corrections, derivatives of the boundary entropy of a region have the form

$$S' = \frac{A'}{4G\hbar} + Q' + S'_{\text{out}} \tag{3.4.26}$$

where Q' are the corrections which start at $O((G\hbar)^0)$. The key point is that Q' is always one order behind A' in the $G\hbar$ perturbation theory. As $G\hbar \to 0$, Q' can only possibly be relevant in situations where $A' = 0$ at $O((G\hbar)^0)$. In this case, $V^i \sim k^i$, and the bulk quantum extremal surface in the vacuum state is a cut of a bulk Killing horizon. But then Q' must be at least $O(G\hbar)$, since $Q' = 0$ on a Killing horizon for any higher derivative theory. Thus we find Eq. (3.4.15) is unchanged at the leading nontrivial order in $G\hbar$.

Higher derivative terms in the bulk action will also modify the definition of the boundary stress tensor. The appearance of the stress tensor in the QNEC and QHANEC proofs comes from the fact that it appears at $O(z^d)$ in the near-boundary expansion of the bulk metric [96]. Higher derivative terms will modify the coefficient of T_{ij} in this expansion, and therefore in the QNEC and QHANEC. (They won't affect the structure of lower-order terms in the asymptotic metric expansion because there aren't any tensors of appropriate weight besides the flat metric η_{ij} [96]). But the new coefficient will differ from the one in Einstein gravity by the addition of terms containing the higher derivative couplings, which are $1/N$-suppressed relative to the Einstein gravity term, and will thus only contribute to the sub-leading parts of the QNEC and QHANEC. Thus the validity of the inequalities at small $G\hbar$ is unaffected.

Boundary Causality Condition implies the AANEC

The proof of this statement was first described in [93]. We direct interested readers to that paper for more detail. Here we will just sketch the proof and note some similarities to the previous two subsections

As discussed above, the bulk causality condition states that no bulk null curve can beat a boundary null geodesic. In the same way that we took a boundary limit of Wedge to prove the quantum half ANEC, the strategy here is to look at time-like curves that hug the boundary. These curves will come asymptotically close to beating the boundary null geodesic and so in some sense derive the most stringent condition on the geometry.

Expanding the near boundary metric in powers of z, we use holographic renormalization to identify pieces of the metric as the stress tensor

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{dz^2 + \eta_{ij}dx^i dx^j + z^d \gamma_{ij}(z, x^i)dx^i dx^j}{z^2} \tag{3.4.27}$$

where $\gamma_{ij}(0, x^i) = \frac{16\pi G_N}{dL^{d-1}}\langle T_{ij}\rangle$. Using null coordinates on the boundary, we can parameterize the example bulk curve by $u \mapsto (u, V(u), Z(u), y^i = 0)$. One constructs a nearly null, time-like curve that starts and ends on the boundary and imposes time delay. If $Z(-L) = Z(L) = 0$, then the bulk causality condition enforces that $V(L) - V(-L) \geq 0$. For the curve used in [93], the $L \to \infty$ limit turns this inequality directly into the boundary ANEC.

3.5 Discussion

We have identified two constraints on the bulk geometry, *entanglement wedge nesting* (EWN) and the *entanglement wedge causality condition* (EWCC), coming directly from the consistency of subregion duality and entanglement wedge reconstruction. The former implies the latter, and the latter implies the boundary causality condition. Additionally, EWN can be understood as a consequence of the quantum focussing conjecture, and EWCC follows from the generalized second law. Both statements in turn have implications for the strongly-coupled large-N theory living on the boundary: the QNEC and QHANEC, respectively. In this section, we list possible generalizations and extensions to this work.

Unsuppressed higher derivative corrections There is no guarantee that higher derivative terms with un-suppressed coefficients are consistent with our conclusions. In fact, in [33] it was observed that Gauss-Bonnett gravity in AdS with an intermediate-scale coupling violates the BCC, and this fact was used to place constraints on the theory. We have seen that the geometrical conditions EWN and EWCC are fundamental to the proper functioning of the bulk/boundary duality. If it turns out that a higher derivative theory invalidates some of our conclusions, it seems more likely that this would be point to a particular pathology of that theory rather than an inconsistency of our results. It would be interesting if EWN and EWCC could be used to place constraints on higher derivative couplings, in the spirit of [33]. We leave this interesting possibility to future work.

A further constraint from subregion duality Entanglement wedge reconstruction implies an additional property that we have not mentioned. Given two boundary regions A and B that are spacelike separated, $\mathcal{E}(A)$ is spacelike separated from $\mathcal{E}(B)$. This property is actually equivalent to EWN for pure states, but is a separate statement for mixed states. In the latter case, it would be interesting to explore the logical relationships of this property to the constraints in 3.1.

Beyond AdS In this paper we have only discussed holography in asymptotically AdS spacetimes. While the QFC, QNEC, and GSL make no reference to asymptotically AdS spacetimes, EWN and $\mathcal{C} \subseteq \mathcal{E}$ currently only have meaning in this context. One could imagine however that a holographic correspondence with subregion duality makes sense in more general spacetimes — perhaps formulated in terms of a "theory" living on a holographic screen [17, 19, 20]. In this case, we expect analogues of EWN and $\mathcal{C} \subseteq \mathcal{E}$. For some initial steps in this direction, see [127].

Quantum generalizations of other bulk facts from generalized entropy A key lesson of this paper is that classical results in AdS/CFT relying on the null energy condition (NEC) can often be made semiclassical by appealing to powerful properties of the generalized entropy: the quantum focussing conjecture and the generalized second law. We expect this to be more general than the semiclassical proofs of EWN and $\mathcal{C} \subseteq \mathcal{E}$ presented here. Indeed, Wall has shown that the generalized second law implies semiclassical generalizations of many celebrated results in classical general relativity, including the singularity theorem [142]. It would be illuminating to see how general this pattern is, both in and out of AdS/CFT. As an example, it is known that strong subadditivity of holographic entanglement entropy can be violated in spacetimes which don't obey the NEC [32]. It seems likely that the QFC can be used to derive strong subadditivity in cases where the NEC is violated due to quantum effects in the bulk.

Gravitational inequalities from field theory inequalities We have seen that the bulk QFC and GSL, which are semi-classical quantum gravity inequalities, imply their non-gravitational limits on the boundary, the QNEC and QHANEC. But we can regard the bulk as an effective field theory of perturbative quantum gravity coupled to matter, and can consider the QNEC and QHANEC for the bulk matter sector. At least when including linearized backreaction of fields quantized on top of a Killing horizon, the QHANEC implies the GSL [138], and the QNEC implies the QFC [21]. In some sense this, "completes" the logical relations of Fig. 3.1.

Support for the quantum extremal surfaces conjecture The logical structure uncovered in this paper relies heavily on the conjecture that the entanglement wedge should be defined in terms of the surface which extremizes the generalized entropy to one side [47] (as opposed to the area). Perhaps similar arguments could be used to prove this conjecture, or at least find an explicit example where extremizing the area is inconsistent with subregion duality.

Connections to Recent Proofs of the AANEC Recent proofs of the AANEC have illuminated the origin of this statement within field theory [52, 76]. In one proof, the engine of the inequality came from microcausality and reflection positivity. In the other, the proof relied on montonoicity of relative entropy for half spaces. A natural next question would be how these two proofs are related, if at all. Our paper seems to offer at least a partial answer for holographic CFTs. Both the monotonicity of relative entropy and microcausality - in our case the QHANEC and BCC, respectively - are implied by the same thing in the bulk: $\mathcal{C} \subseteq \mathcal{E}$. In 3.2, we gave a motivation for this geometric constraint from subregion duality. It would be interesting to see how the statement of $\mathcal{C} \subseteq \mathcal{E}$ in a purely field theoretic language is connected to both the QHANEC and causality.

Chapter 4

Local Modular Hamiltonians from the Quantum Null Energy Condition

4.1 Introduction and Summary

The reduced density operator ρ for a region in quantum field theory encodes all of the information about observables localized to that region. Given any ρ, one can define the *modular Hamiltonian* K by

$$\rho = e^{-K}. \tag{4.1.1}$$

Knowledge of this operator is equivalent to knowledge of ρ, but the modular Hamiltonian frequently appears in calculations involving entanglement entropy. In general, i.e. for arbitrary states reduced to arbitrary regions, K is a complicated non-local operator. However, in certain cases it is known to simplify.

The most basic example where K simplifies is the vacuum state of a QFT in Rindler space, i.e. the half-space $t = 0, x \geq 0$. The Bisognano–Wichmann theorem [16] states that in this case the modular Hamiltonian is

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_0^\infty x\, T_{tt}\, dx \tag{4.1.2}$$

where $\Delta K \equiv K - \langle K \rangle_{\text{vac}}$ defines the vacuum-subtracted modular Hamiltonian, and y are $d-2$ coordinates parametrizing the transverse directions. The vacuum subtraction generally removes regulator-dependent UV-divergences in K. Other cases where the modular Hamiltonian is known to simplify to an integral of local operators are obtained via conformal transformation of Eq. (4.1.2), including spherical regions in CFTs [37], regions in a thermal state of 1+1 CFTs [35], and null slabs [25, 22].

Using conservation of the energy-momentum tensor, one can easily re-express the Rindler modular Hamiltonian in Eq. (4.1.2) as an integral over the future Rindler horizon $u \equiv t - x =$

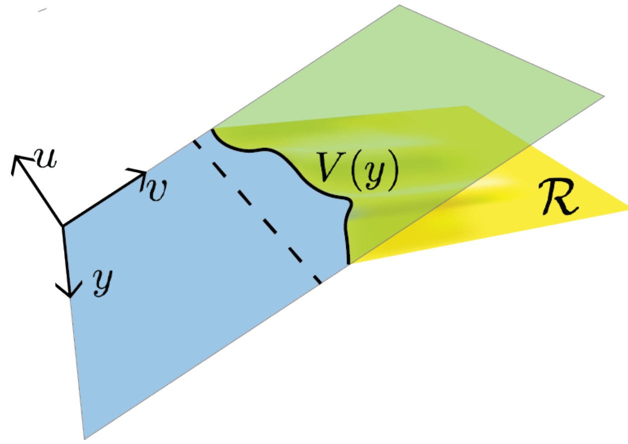

Figure 4.1: This image depicts a section of the plane $u = t - x = 0$. The region $\mathcal{R}$ is defined to be one side of a Cauchy surface split by the codimension-two entangling surface $\partial\mathcal{R} = \{(u = 0, v = V(y), y)\}$. The dashed line corresponds to a flat cut of the null plane.

0 which bounds the future of the Rindler wedge:

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_0^\infty v\, T_{vv}\, dv, \tag{4.1.3}$$

where $v \equiv t + x$. It is important to note that standard derivations of (4.1.2) or (4.1.3), e.g. [16, 37], do not apply when the entangling surface is defined by a non-constant cut of the Rindler horizon (see Fig. 4.1). One of the primary goals of this paper is to provide such a derivation.

For a large class of quantum field theories satisfying a precise condition specified momentarily, we will show that the vacuum modular Hamiltonian for the region $\mathcal{R}[V(y)]$ above an arbitrary cut $v = V(y)$ of a null plane is given by

$$\Delta K = \frac{2\pi}{\hbar} \int d^{d-2}y \int_{V(y)}^\infty (v - V(y))\, T_{vv}\, dv \tag{4.1.4}$$

This equation has been previously derived by Wall for free field theories [138] building on [29, 131], and to linear order in the deformation away from $V(y) = const$ in general QFTs by Faulkner et al. [52]. In CFTs, conformal transformations of Eq. (4.1.4) yield versions of the modular Hamiltonian for non-constant cuts of the causal diamond of a sphere.

The condition leading to Eq. (4.1.4) is that the theory should satisfy the *quantum null energy condition* (QNEC) [21, 26, 96, 4] — an inequality between the stress tensor and the von Neumann entropy of a region — and saturate the QNEC in the vacuum for regions defined by cuts of a null plane. We will review the statement of the QNEC in Sec. 4.2.

The QNEC has been proven for free and superrenormalizable [26], as well as holographic [96, 4] quantum field theories. We take this as reasonable evidence that the QNEC is a true fact about relativistic quantum field theories in general, and for the purposes of this paper take it as an assumption. In Sec. 4.2 we will show how saturation of the QNEC in a given state leads to an operator equality relating certain derivatives of the modular Hamiltonian of that state to the energy-momentum tensor. Applied to the case outlined above, this operator equality will be integrated to give Eq. (4.1.4).

Given the argument in Sec. 4.2, the only remaining question is whether the QNEC is in fact saturated in the vacuum state for entangling surfaces which are cuts of a null plane. This has been shown for free theories in [26]. In Sec. 4.3, we prove that this is the case for holographic theories to all orders in $1/N$. We emphasize that Eq. (4.1.4) holds purely as a consequence of the validity of the QNEC and the saturation in the vacuum for $\mathcal{R}$, two facts which are potentially true in quantum field theories much more generally than free and holographic theories.

Finally, in Sec. 4.4 we will conclude with a discussion of possible extensions to curved backgrounds and more general regions, connections between the relative entropy and the QNEC, and relations to other work.

4.2 Main Argument

Review of QNEC

The von Neumann entropy of a region in quantum field theory can be regarded as a functional of the entangling surface. We will primarily be interested in regions to one side of a cut of a null plane in flat space, for which the entangling surface can be specified by a function $V(y)$ which indicates the v-coordinate of the cut as a function of the transverse coordinates, collectively denoted y. See Fig. 4.1 for the basic setup. Each cut $V(y)$ defines a half-space, namely the region to one side of the cut. We will pick the side towards the future of the null plane. For the purposes of this section we are free to consider the more general situation where the entangling surface is only *locally* given by a cut of a null plane. Thus the von Neumann entropy can be considered as a functional of a profile $V(y)$ which defines the shape of the entangling surface, at least locally.

Suppose we define a one-parameter family of cuts $V(y;\lambda) \equiv V(y;0)+\lambda\dot{V}(y)$, with $\dot{V}(y) > 0$ to ensure that $\mathcal{R}(\lambda_1) \subset \mathcal{R}(\lambda_2)$ if $\lambda_1 > \lambda_2$. If $S(\lambda)$ is the entropy of region $\mathcal{R}(\lambda)$, then the QNEC in integrated form states that

$$\int d^{d-2}y \; \langle T_{vv}(y)\rangle \, \dot{V}(y)^2 \geq \frac{\hbar}{2\pi}\frac{d^2S}{d\lambda^2}. \tag{4.2.1}$$

In general there would be a $\sqrt{h}$ induced metric factor weighting the integral, but here and in the rest of the paper we will assume that the y coordinates have been chosen such that $\sqrt{h} = 1$.

By taking advantage of the arbitrariness of $\dot{V}(y)$ we can derive from this the local form of the QNEC. If we take a limit where $\dot{V}(y')^2 \to \delta(y-y')$, then the l.h.s. reduces to $\langle T_{vv}\rangle$.

We *define* $S''(y)$ as the limit of $d^2S/d\lambda^2$ in the same situation:

$$\frac{d^2S}{d\lambda^2} \to S''(y) \quad \text{when} \quad \dot{V}(y')^2 \to \delta(y-y'). \tag{4.2.2}$$

Taking the limit of the integrated QNEC then gives:

$$\langle T_{vv} \rangle \geq \frac{\hbar}{2\pi} S''. \tag{4.2.3}$$

The QNEC in (4.2.3) together with strong subadditivity can likewise be used to go backward and derive the integrated QNEC (4.2.1) [21, 26, 96]. The details of that argument are not important here. In the next section we will discuss the consequences of the saturation of the QNEC, and will have to distinguish whether we mean saturation of the nonlocal inequality Eq. (4.2.1) or the local inequality Eq. (4.2.3), the latter condition being weaker.

The QNEC under state perturbations

In this section we consider how the QNEC behaves under small deformations of the state. We begin with a reference state σ and consider the deformed state $\rho = \sigma + \delta\rho$, with $\delta\rho$ traceless but otherwise arbitrary.

Consider a one-parameter family of regions $\mathcal{R}(\lambda)$ as in the previous section. Define $\overline{\mathcal{R}}(\lambda)$ to be the complement of $\mathcal{R}(\lambda)$ within a Cauchy surface. The reduced density operator for any given region $\mathcal{R}(\lambda)$ given by

$$\rho(\lambda) = \sigma(\lambda) + \delta\rho(\lambda) = \mathrm{Tr}_{\overline{\mathcal{R}}(\lambda)}\sigma + \mathrm{Tr}_{\overline{\mathcal{R}}(\lambda)}\delta\rho. \tag{4.2.4}$$

By the First Law of entanglement entropy, the entropy of $\rho(\lambda)$ is given by

$$S(\rho(\lambda)) = S(\sigma(\lambda)) - \mathrm{Tr}_{\mathcal{R}(\lambda)}\delta\rho(\lambda)\log\sigma(\lambda) + o(\delta\rho^2). \tag{4.2.5}$$

The second term can be written in a more useful way be defining the modular Hamiltonian $K_\sigma(\lambda)$ as

$$K_\sigma(\lambda) \equiv -\mathbb{1}_{\overline{\mathcal{R}}(\lambda)} \otimes \log\sigma(\lambda). \tag{4.2.6}$$

Defining $K_\sigma(\lambda)$ this way makes it a global operator, which makes taking derivatives with respect to λ formally simpler. Using this definition, we can write Eq. (4.2.5) as

$$S(\rho(\lambda)) = S(\sigma(\lambda)) + \mathrm{Tr}\,\delta\rho K_\sigma(\lambda) + o(\delta\rho^2). \tag{4.2.7}$$

Now in the second term the trace is over the global Hilbert space, and the λ-dependence has been isolated to the operator $K_\sigma(\lambda)$. Taking two derivatives, and simplifying the notation slightly, we find

$$\frac{d^2S}{d\lambda^2}(\rho) = \frac{d^2S}{d\lambda^2}(\sigma) + \mathrm{Tr}\,\delta\rho\frac{d^2K_\sigma}{d\lambda^2} + o(\delta\rho^2). \tag{4.2.8}$$

Suppose that the integrated QNEC, Eq. (4.2.1), is saturated in the state σ for all profiles $\dot{V}(y)$. Then, using Eq. (4.2.8), the integrated QNEC for the state ρ can be written as

$$\int d^{d-2}y \; (\mathrm{Tr}\, \delta\rho\, T_{vv})\, \dot{V}^2 \geq \frac{\hbar}{2\pi} \mathrm{Tr}\, \delta\rho \frac{d^2 K_\sigma}{d\lambda^2} + o(\delta\rho^2). \tag{4.2.9}$$

The operator $\delta\rho$ was arbitrary, and in particular could be replaced by $-\delta\rho$. Then the only way that Eq. 4.2.9 can hold is if we have the operator equality

$$\frac{d^2 K_\sigma}{d\lambda^2} = C + \frac{2\pi}{\hbar} \int d^{d-2}y \, T_{vv} \dot{V}^2. \tag{4.2.10}$$

Here C is a number that we cannot fix using this method that is present because of the tracelessness of $\delta\rho$.

Eq. (4.2.10) can be integrated to derive the full modular Hamiltonian K_σ if we have appropriate boundary conditions. Up until now we have only made use of local properties of the entangling surface, but in order to provide boundary conditions for the integration of Eq. (4.2.10) we will assume that the entangling surface is globally given by a cut of a null plane, and that $V(y; \lambda = 0) = 0$. We will also make σ the vacuum state. In that situation it is known that the QNEC is saturated for free theories, and in the next section we will show that this is also true for holographic theories at all orders in the large-N expansion.

Our first boundary condition is at $\lambda = \infty$.[1] Since we expect that $K_\sigma(\lambda)$ should have a finite expectation value in any state as $\lambda \to \infty$, it must be that $dK_\sigma/d\lambda \to 0$ as $\lambda \to \infty$. Then integrating Eq. (4.2.10) gives

$$\frac{dK_\sigma}{d\lambda} = -\frac{2\pi}{\hbar} \int d^{d-2}y \int_{V(y;\lambda)}^{\infty} dv \, T_{vv} \dot{V}. \tag{4.2.11}$$

Note that this equation implies that the vacuum expectation value $\langle K_\sigma(\lambda)\rangle_{vac}$ is actually λ-independent, which makes vacuum subtraction easy.

Our second boundary condition is Eq. (4.1.3), valid at $\lambda = 0$ when $V(y; \lambda) = 0$. Integrating once morenand making use of this boundary condition, we find

$$\Delta K_\sigma(\lambda) = \frac{2\pi}{\hbar} \int d^{d-2}y \int_{V(y;\lambda)}^{\infty} (v - V(y; \lambda))\, T_{vv} \, dv \tag{4.2.12}$$

which is Eq. (4.1.4). Note that the l.h.s. of this equation is now the vacuum-subtracted modular Hamiltonian.

[1] It is not always possible to consider the $\lambda \to \infty$ limit of a null perturbation to an entangling surface because parts of the entangling surface may become timelike related to each other at some finite value of λ, at which point the surface is no longer the boundary of a region on a Cauchy surface. However, when the entangling surface is globally equal to a cut of a null plane this is not an issue.

Before moving on, we will briefly comment on the situation where the QNEC in Eq. (4.2.3), is saturated but the integrated QNEC, Eq. (4.2.1), is not. Then, analogously to S'' in Eq. (4.2.2), one may define a local second derivative of K_σ:

$$\frac{d^2 K_\sigma}{d\lambda^2} \to K''_\sigma(y) \quad \text{when} \quad \dot{V}(y')^2 \to \delta(y-y'). \tag{4.2.13}$$

Very similar manipulations then show that saturation of (4.2.3) implies the equality

$$K''_\sigma = \frac{2\pi}{\hbar} T_{vv}. \tag{4.2.14}$$

This equation is weaker than Eq. (4.2.10), which is meant to be true for arbitrary profiles of $\dot{V}(y)$, but it may have a greater regime of validity. We will comment on this further in Sec. 4.4.

4.3 Holographic Calculation

In the previous section we argued that the form of the modular Hamiltonian could be deduced from saturation of the QNEC. In this section we will use the holographic entanglement entropy formula [126, 125, 86, 50] to show that the QNEC is saturated in vacuum for entangling surfaces defined by arbitrary cuts $v = V(y)$ of the null plane $u = 0$ in holographic theories. Our argument applies to any holographic theory defined by a relevant deformation to a holographic CFT, and will be at all orders in the large-N expansion. To reach arbitrary order in $1/N$ we will assume that the all-orders prescription for von Neumann entropy is given by the quantum extremal surface proposal of Engelhardt and Wall [47]. This is the same context in which the holographic proof of the QNEC was extended to all orders in $1/N$ [4].[2]

As before, the entangling surface in the field theory is given by the set of points $\partial\mathcal{R} = \{(u,v,y) : v = V(y), u = 0\}$ with null coordinates $u = t-x$ and $v = t+x$, and the region $\mathcal{R}$ is chosen to lie in the $u < 0$ portion of spacetime. Here y represents $d-2$ transverse coordinates. The bulk quantum extremal surface anchored to this entangling surface is parameterized by the functions $\bar{V}(y,z)$ and $\bar{U}(y,z)$. It was shown in [96, 4] that if we let the profile $V(y)$ depend on a deformation parameter λ, then the second derivative of the entropy is given by

$$\frac{d^2 S}{d\lambda^2} = -\frac{d}{4G\hbar} \int d^{d-2}y \, \frac{d\bar{U}_{(d)}}{d\lambda} \,, \tag{4.3.1}$$

to all orders in $1/N$, where $\bar{U}_{(d)}(y)$ is the coefficient of z^d in the small-z expansion of $\bar{U}(z,y)$. We will show that $\bar{U} = 0$ identically for any profile $V(y)$, which then implies that $d^2S/d\lambda^2 = 0$, which is the statement of QNEC saturation in the vacuum.

[2] It is crucial that we demonstrate saturation beyond leading order in large-N. The argument in the previous section used *exact* saturation, and an error that is naïvely subleading when evaluated in certain states may become very large in others.

One way to show that $\bar{U}$ vanishes is to demonstrate that $\bar{U} = 0$ solves the quantum extremal surface equations of motion in the bulk geometry dual to the vacuum state of the boundary theory. The quantum extremal surface is defined by having the sum of the area plus the bulk entropy on one side be stationary with respect to first-order variations of its position. One can show that $\bar{U} = 0$ is a solution to the equations of motion if any only if

$$\frac{\delta S_{\text{bulk}}}{\delta \bar{V}(y,z)} = 0 \tag{4.3.2}$$

in the vacuum everywhere along the extremal surface. This would follow from null quantization if the bulk fields were free [26], but that would only allow us to prove the result at order-one in the $1/N$ expansion.

For an all-orders argument, we opt for a more indirect approach using subregion duality, or entanglement wedge reonstruction [42, 81, 44, 72].[3] A version of this argument first appeared in [4], and we elaborate on it here.

Entanglement wedge reconstruction requires two important consistency conditions in the form of constraints on the bulk geometry which must hold at all orders in $1/N$: The first constraint, *entanglement wedge nesting* (EWN), states that if one boundary region is contained inside the domain of dependence of another, then the quantum extremal surface associated to the first boundary region must be contained within the entanglement wedge of the second boundary region [42, 140]. The second constraint, $\mathcal{C} \subseteq \mathcal{E}$, demands that the causal wedge of a boundary region be contained inside the entanglement wedge of that region [42, 81, 140, 47, 85]. Equivalently, it says that no part of the quantum extremal surface of a given boundary region can be timelike-related to the (boundary) domain of dependence of that boundary region. It was shown in [4] that $\mathcal{C} \subseteq \mathcal{E}$ follows from EWN, and EWN itself is simply the statement that a boundary region should contain all of the information about any of its subregions. We will now explain the consequences of these two constraints for $\bar{U}(y,z)$.

Without loss of generality, suppose the region $\mathcal{R}$ is defined by a coordinate profile which is positive, $V(y) > 0$. Consider a second region $\mathcal{R}_0$ which has an entangling surface at $v = u = 0$ and whose domain of dependence (i.e., Rindler space) contains $\mathcal{R}$. The quantum extremal surface associated to $\mathcal{R}_0$ is given by $\bar{U}_0 = \bar{V}_0 = 0$. This essentially follows from symmetry.[4] The entanglement wedge of $\mathcal{R}_0$ is then a bulk extension of the boundary Rindler space, namely the set of bulk points satisfying $u \leq 0$ and $v \geq 0$. Then EWN implies that $\bar{U} \leq 0$ and $\bar{V} \geq 0$.

The only additional constraint we need from $\mathcal{C} \subseteq \mathcal{E}$ is the requirement that the quantum extremal surface for $\mathcal{R}$ not be in the past of the domain of dependence of $\mathcal{R}$. From the definition of $\mathcal{R}$, it is clear that a bulk point is in the past of the domain of dependence of $\mathcal{R}$ if and only if it is in the past of the region $u < 0$ on the boundary, which is the same as the

[3]The entanglement wedge of a boundary region is the set of bulk points which are spacelike- or null-related to that region's quantum extremal surface on the same side of the quantum extremal surface as the boundary region itself.

[4]One might worry that the quantum extremal surface equations display spontaneous symmetry breaking in the vacuum, but this can be ruled out using $\mathcal{C} \subseteq \mathcal{E}$ with an argument similar to the one we present here.

region $u < 0$ in the bulk. Therefore it must be that $\bar{U} \geq 0$. Combined with the constraint from EWN above, we then conclude that the only possibility is $\bar{U} = 0$. This completes the proof that the QNEC is saturated to all orders in $1/N$. The saturation of the integrated QNEC in the vacuum in particular implies that strong subadditivity is saturated for regions on the null plane [21, 26, 96].

4.4 Discussion

We conclude by discussing the generality of our analysis, some implications and future directions, and connections with previous work.

Generalizations and Future Directions

General Killing horizons Though we restricted to cuts of Rindler horizons in flat space for simplicity, all of our results continue to hold for cuts of bifurcate Killing horizons for QFTs defined in arbitrary spacetimes, assuming the QNEC is true and saturated in the vacuum in this context. In particular, Eq. (4.1.4) holds with v a coordinate along the horizon. For holographic theories, entanglement wedge nesting (EWN) and the entanglement wedge being outside of the causal wedge ($\mathcal{C} \subset \mathcal{E}$) continue to prove saturation of the QNEC. To see this, note that a Killing horizon on the boundary implies a corresponding Killing horizon in the bulk. Now take the reference region $\mathcal{R}_0$ satisfying $V(y) = U(y) = 0$ to be the boundary bifurcation surface. By symmetry, the associated quantum extremal surface lies on the bifurcation surface of the bulk Killing horizon. Then the quantum extremal surface of the region $\mathcal{R}$ defined by $V(y) \geq 0$ must lie in the entanglement wedge of $\mathcal{R}_0$ — inside the bulk horizon — by entanglement wedge nesting, but must also lie on or outside of the bulk horizon by $\mathcal{C} \subset \mathcal{E}$. Thus it lies on the bulk horizon, $\bar{U} = 0$, and the QNEC remains saturated by Eq. (4.3.1).

Future work In this work, we have only established the form of $K_{\mathcal{R}}$ for regions $\mathcal{R}$ bounded by arbitrary cuts of a null plane. A natural next direction would be to understand if and how we can extend Eq. (4.2.14) to more general entangling surfaces. As discussed above, the QNEC was shown to hold for locally flat entangling surfaces in holographic, free and super-renormalizable field theories [26, 96, 4]. Thus, if we could prove saturation, i.e. that $S''_{\text{vac}} = 0$ at all orders in $1/N$, then we would establish (4.2.14) for all regions with a locally flat boundary.

One technique to probe this question is to perturb the entangling surface away from a flat cut and compute the contributions to the QNEC order-by-order in a perturbation parameter ϵ. After the completion of this work, the result of [106] showed that for holographic theories $S''_{\text{vac}}(y) = 0$ as long as the entangling surface is stationary in a neighborhood of y.

Another interesting problem is to show that in a $g$eneral QFT vacuum, null derivatives of entanglement entropy across arbitrary cuts of null planes vanish. That, along with a general proof of QNEC will establish (18) as a consequence. We will leave this to future work.

The QNEC as $S(\rho\|\sigma)'' \geq 0$

There is a connection between the QNEC and relative entropy, first pointed out in [4], that we elaborate on here. The relative entropy $S(\rho\|\sigma)$ between two states ρ and σ is defined as

$$S(\rho|\sigma) = Tr\left(\rho\log\rho\right) - Tr\left(\rho\log\sigma\right) \tag{4.4.1}$$

and provides a measure of distinguishability between the two states [117]. Substituting the definition of K, Eq. (4.1.1), into Eq. (4.4.1) provides a useful alternate presentation:

$$S(\rho\|\sigma) = \langle K_\sigma\rangle_\rho - S(\rho). \tag{4.4.2}$$

If Eq. (4.1.4) is valid, then taking two derivatives with respect to a deformation parameter, as in the main text, shows that the integrated QNEC, Eq. (4.2.1), is equivalent to

$$\partial_\lambda^2 S(\rho(\lambda)\|\sigma(\lambda)) \geq 0. \tag{4.4.3}$$

For comparison, monotonicity of relative entropy for the types of regions and deformations we have been discussing can be written as

$$\partial_\lambda S(\rho(\lambda)\|\sigma(\lambda)) \leq 0. \tag{4.4.4}$$

Eq. (4.4.3) is a sort of "convexity" of relative entropy.[5] Unlike monotonicity of relative entropy, which says that the first derivative is non-positive, there is no general information-theoretic reason for the *second* derivative to be non-negative. In the event that Eq. (4.2.14) holds but not Eq. (4.1.4), we would still have

$$S(\rho\|\sigma)'' \geq 0. \tag{4.4.5}$$

where the $''$ notation denotes a local deformation as in Sec. 4.2.

It would be extremely interesting to characterize what about quantum field theory and null planes makes (4.4.3) true. We can model the null deformation as a non-unitary time evolution in the space of states, with the vacuum state serving as an equilibrium state for this evolution. Then an arbitrary finite-energy state will relax toward the equilibrium state, with the relative entropy $S(\rho\|\sigma)$ characterizing the free energy as a function of time. Monotonicity of relative entropy is then nothing more than the statement that free energy decreases, i.e. the second law of thermodynamics. The second derivative statement gives more information about the approach to equilibrium. If that approach is of the form of

[5] This is distinct from the well-known convexity of relative entropy, which says that $S(t\rho_1+(1-t)\rho_2\|\sigma) \leq tS(\rho_1\|\sigma) + (1-t)S(\rho_2\|\sigma)$.

exponential decay, then all successive derivatives would alternate in sign. However, for null deformations in quantum field theory we do not expect to have a general bound on the behavior of derivatives of the energy-momentum tensor, meaning that the third derivative of the free energy should not have a definite sign.[6] Perhaps there is some way of characterizing the approach to equilibrium we have here, which is in some sense smoother than the most general possibility but not so constrained as to force exponential behavior.

Relation to previous work

Faulkner, Leigh, Parrikar and Wang [52] have discussed results very similar to the ones presented here. They demonstrated that for first-order null deformations $\delta V(y)$ to a flat cut of a null plane, the perturbation to the modular Hamiltonian takes the form

$$\langle K_{\mathcal{R}}\rangle_\psi - \langle K_{\mathcal{R}_0}\rangle_\psi = -\frac{2\pi}{\hbar}\int d^{d-2}y \int_{V(y)} dv\, T_{vv}(y)\,\delta V(y) \tag{4.4.6}$$

This is precisely the form expected from our equation (4.1.4). Faulkner et al. went on to suggest that the natural generalization of the modular Hamiltonian to finite deformations away from a flat cut takes the form of Eq. (4.1.4). In the context of holography they showed that this conclusion applied both on the boundary and in the bulk is consistent with JLMS [92]. In the present paper, we have shown that Eq. (4.1.4) holds for theories which obey the QNEC, and for which the QNEC is saturated in the vacuum. A non-perturbative, field theoretic proof of these assumptions remains a primary goal of future work.

[6]We thank Aron Wall for a discussion of this point.

Chapter 5

The Quantum Null Energy Condition, Entanglement Wedge Nesting, and Quantum Focusing

The Quantum Focusing Conjecture (QFC) is a new principle of semiclassical quantum gravity proposed in [21]. Its formulation is motivated by classical focusing, which states that the expansion θ of a null congruence of geodesics is nonincreasing. Classical focusing is at the heart of several important results of classical gravity [120, 77, 78, 60], and likewise quantum focusing can be used to prove quantum generalizations of many of these results [141, 142, 18, 3].

One of the most important and surprising consequences of the QFC is the Quantum Null Energy Condition (QNEC), which was discovered as a particular nongravitational limit of the QFC [21]. Subsequently the QNEC was proven for free fields [26] and for holographic CFTs on flat backgrounds [96] (and recently extended in [62] in a similar way as we do here). The formulation of the QNEC which naturally comes out of the proofs we provide here is as follows.

Consider a codimension-two Cauchy-splitting surface Σ, which we will refer to as the entangling surface. The Von Neumann entropy $S[\Sigma]$ of the interior (or exterior) of Σ is a functional of Σ, and in particular is a functional of the embedding functions $X^i(y)$ that define Σ. Choose a one-parameter family of deformed surfaces $\Sigma(\lambda)$, with $\Sigma(0) = \Sigma$, such that (i) $\Sigma(\lambda)$ is given by flowing along null geodesics generated by the null vector field k^i normal to Σ for affine time λ , and (ii) $\Sigma(\lambda)$ is either "shrinking" or "growing" as a function of λ, in the sense that the domain of dependence of the interior of Σ is either shrinking or growing. Then for any point on the entangling surface we can define the combination

$$T_{ij}(y)k^i(y)k^j(y) - \frac{1}{2\pi}\frac{d}{d\lambda}\left(\frac{k^i(y)}{\sqrt{h(y)}}\frac{\delta S_{ren}}{\delta X^i(y)}\right). \tag{5.0.1}$$

Here $\sqrt{h(y)}$ is the induced metric determinant on Σ. Writing this down in a general curved background requires a renormalization scheme both for the energy-momentum tensor T_{ij} and the renormalized entropy S_{ren}. Assuming that this quantity is scheme-independent (and hence well-defined), the QNEC states that it is positive. Our main task is to determine the necessary and sufficient conditions we need to impose on Σ and the background spacetime at the point y in order that the QNEC hold.

In addition to a proof through the QFC, the holographic proof method of [96] is easily adaptable to answering this question in full generality. The backbone of that proof is Entanglement Wedge Nesting (EWN), which is a consequence of subregion duality in AdS/CFT [3]. A given region on the boundary of AdS is associated with a particular region of the bulk, called the entanglement wedge, which is defined as the bulk region spacelike-related to the extremal surface [126, 86, 47, 45] used to compute the CFT entropy on the side toward the boundary region. This bulk region is dual to the given boundary region, in the sense that there is a correspondence between the algebra of operators in the bulk region and that of the operators in the boundary region which are good semiclassical gravity operators (i.e., they act within the subspace of semiclassical states) [42, 92, 44]. EWN is the statement that nested boundary regions must be dual to nested bulk regions, and clearly follows from the consistency of subregion duality.

While the QNEC can be derived from both the QFC and EWN, there has been no clear connection between these derivations.[1] As it stands, there are apparently two QNECs, the QNEC-from-QFC and the QNEC-from-EWN. We will show in full generality that these two QNECs are in fact the same, at least in $d \leq 5$ dimensions.

Here is a summary of our results:

- The holographic proof of the QNEC from EWN is extended to CFTs on arbitrary curved backgrounds. In $d = 5$ we find that the necessary and sufficient conditions for the ordinary QNEC to hold at a point are that[2]

$$\theta_{(k)} = \sigma_{ab}^{(k)} = D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = R_{ka} = 0 \tag{5.0.2}$$

  at that point. For $d < 5$ only a subset of these conditions are necessary. This is the subject of §5.1.

- We also show holographically that under the weaker set of conditions

$$\sigma_{ab}^{(k)} = D_a\theta_{(k)} + R_{ka} = D_a\sigma_{bc}^{(k)} = 0 \tag{5.0.3}$$

  the Conformal QNEC holds. The Conformal QNEC was introduced in [96] as a conformally-transformed version of the QNEC. This is the strongest inequality that we can get out of EWN. This is the subject of §5.1

[1] In [3] it was shown that the QFC in the bulk implies EWN, which in turn implies the QNEC. This is not the same as the connection we are referencing here. The QFC which would imply the boundary QNEC in the sense that we mean is a *boundary* QFC, obtained by coupling the boundary theory to gravity.

[2] Here $\sigma_{ab}^{(k)}$ and $\theta_{(k)}$ are the shear and expansion in the k^i direction, respectively, and D_a is a surface covariant derivative. Our notation is further explained in Appendix A.

- By taking the non-gravitational limit of the QFC we are able to derive the QNEC again under the same set of conditions as we did for EWN. This is the subject of §5.2.
- We argue in §5.2 that the statement of the QNEC is scheme-independent whenever the conditions that allow us to prove it hold. This shows that the two proofs of the QNEC are actually proving the same, unambiguous field–theoretic bound.

We conclude in §5.3 with a discussion and suggest future directions. A number of technical Appendices are included as part of our analysis.

Relation to other work While this work was in preparation, [62] appeared which has overlap with our discussion of EWN and the scheme-independence of the QNEC. The results of [62] relied on a number of assumptions about the background: the null curvature condition and a positive energy condition. From this they derive certain sufficient conditions for the QNEC to hold. We do not assume anything about our backgrounds a priori, and include all relevant higher curvature corrections. This gives our results greater generality, as we are able to find both necessary and sufficient conditions for the QNEC to hold.

5.1 Entanglement Wedge Nesting

Subregion Duality

The statement of AdS/CFT includes a correspondence between operators in the semiclassical bulk gravitational theory and CFT operators on the boundary. Moreover, it has been shown [72, 44] that such a correspondence exists between the operator algebras of subregions in the CFT and certain associated subregions in the bulk as follows: Consider a spatial subregion A in the boundary geometry. The extremal surface anchored to ∂A, which is used to compute the entropy of A [126, 86], bounds the so-called entanglement wedge of A, $\mathcal{E}(A)$, in the bulk. More precisely $\mathcal{E}(A)$ is the codimension-zero bulk region spacelike-related to the extremal surface on the same side of the extremal surface as A. Subregion duality is the statement that the operator algebras of $\mathcal{D}(A)$ and $\mathcal{E}(A)$ are dual, where $\mathcal{D}(A)$ denotes the domain of dependence of A.

Entanglement Wedge Nesting The results of this section follow from EWN, which we now describe. Consider two boundary regions A_1 and A_2 such that $\mathcal{D}(A_1) \subseteq \mathcal{D}(A_2)$. Then consistency of subregion duality implies that $\mathcal{E}(A_1) \subseteq \mathcal{E}(A_2)$ as well, and this is the statement of EWN. In particular, EWN implies that the extremal surfaces associated to A_1 and A_2 cannot be timelike-related.

We will mainly be applying EWN to the case of a one-paramter family of boundary regions, $A(\lambda)$, where $\mathcal{D}(A(\lambda_1)) \subseteq \mathcal{D}(A(\lambda_2))$ whenever $\lambda_1 \leq \lambda_2$. Then the union of the one-parameter family of extremal surfaces associated to $A(\lambda)$ forms a codimension-one surface

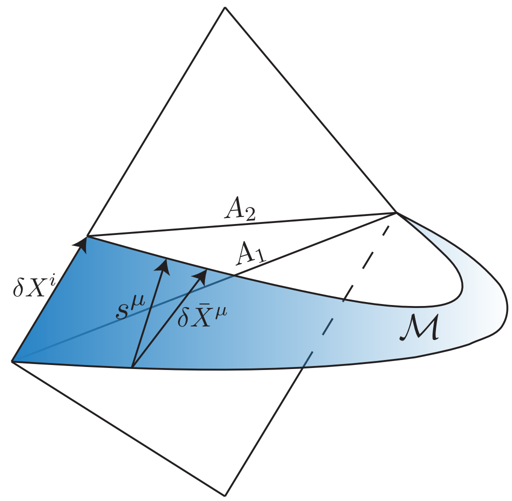

Figure 5.1: Here we show the holographic setup which illustrates Entanglement Wedge Nesting. A spatial region A_1 on the boundary is deformed into the spatial region A_2 by the null vector δX^i. The extremal surfaces of A_1 and A_2 are connected by a codimension-one bulk surface $\mathcal{M}$ (shaded blue) that is nowhere timelike by EWN. Then the vectors $\delta\bar{X}^\mu$ and s^μ, which lie in $\mathcal{M}$, have nonnegative norm.

in the bulk that is nowhere timelike. We denote this codimension-one surface by $\mathcal{M}$. See Fig. 5.1 for a picture of the setup.

Since $\mathcal{M}$ is nowhere timelike, every one of its tangent vectors must have nonnegative norm. In particular, consider the embedding functions $\bar{X}^\mu$ of the extremal surfaces in some coordinate system. Then the vectors $\delta\bar{X}^\mu \equiv \partial_\lambda \bar{X}^\mu$ is tangent to $\mathcal{M}$, and represents a vector that points from one extremal surface to another. Hence we have $(\delta\bar{X})^2 \geq 0$ from EWN, and this is the inequality that we will discuss for most of the remainder of this section.

Before moving on, we will note that $(\delta\bar{X})^2 \geq 0$ is not necessarily the strongest inequality we get from EWN. At each point on $\mathcal{M}$, the vectors which are tangent to the extremal surface passing through that point are known to be spacelike. Therefore if $\delta\bar{X}^\mu$ contains any components which are tangent to the extremal surface, they will serve to make the inequality $(\delta\bar{X})^2 \geq 0$ weaker. We define the vector s^μ at any point of $\mathcal{M}$ to be the part of $\delta\bar{X}^\mu$ orthogonal to the extremal surface passing through that point. Then $(\delta\bar{X})^2 \geq s^2 \geq 0$. We will discuss the $s^2 \geq 0$ inequality in §5.1 after handling the $(\delta\bar{X})^2 \geq 0$ case.

Near-Boundary EWN

In this section we explain how to calculate the vector $\delta\bar{X}^\mu$ and s^μ near the boundary explicitly in terms of CFT data. Then the EWN inequalities $(\delta\bar{X})^2 > 0$ and $s^2 > 0$ can be given a CFT meaning. The strategy is to use a Fefferman-Graham expansion of both the metric and extremal surface, leading to equations for $\delta\bar{X}^\mu$ and s^μ as power series in the bulk coordinate

z (including possible log terms). In the following sections we will analyze the inequalities that are derived in this section.

Bulk Metric We work with a bulk theory in AdS_{d+1} that consists of Einstein gravity plus curvature-squared corrections. For $d \leq 5$ this is the complete set of higher curvature corrections that have an impact on our analysis. The Lagrangian is[3]

$$\mathcal{L} = \frac{1}{16\pi G_N}\left(\frac{d(d-1)}{\tilde{L}^2} + \mathcal{R} + \ell^2\lambda_1\mathcal{R}^2 + \ell^2\lambda_2\mathcal{R}_{\mu\nu}^2 + \ell^2\lambda_{GB}\mathcal{L}_{GB}\right), \tag{5.1.1}$$

where $\mathcal{L}_{GB} = \mathcal{R}_{\mu\nu\rho\sigma}^2 - 4\mathcal{R}_{\mu\nu}^2 + \mathcal{R}^2$ is the Gauss–Bonnet Lagrangian, ℓ^2 is the cutoff scale, and $\tilde{L}^2$ is the scale of the cosmological constant. The bulk metric has the following near boundary expansion in Fefferman-Graham gauge [74]:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \bar{g}_{ij}(x,z)dx^i dx^j), \tag{5.1.2}$$

$$\bar{g}_{ij}(x,z) = g_{ij}^{(0)}(x) + z^2 g_{ij}^{(2)}(x) + z^4 g_{ij}^{(4)}(x) + \ldots + z^d \log z\, g_{ij}^{(d,log)}(x) + z^d g_{ij}^{(d)}(x) + o(z^d). \tag{5.1.3}$$

Note that the length scale L is different from $\tilde{L}$, but the relationship between them will not be important for us. Demanding that the above metric solve bulk gravitational equations of motion gives expressions for all of the $g_{ij}^{(n)}$ for $n < d$, including $g_{ij}^{(d,log)}(x)$, in terms of $g_{ij}^{(0)}(x)$. This means, in particular, that these terms are all state-independent. One finds that $g_{ij}^{(d,log)}(x)$ vanishes unless d is even. We provide explicit expressions for some of these terms in Appendix C.

The only state-dependent term we have displayed, $g_{ij}^{(d)}(x)$, contains information about the expectation value of the energy-momentum tensor T_{ij} of the field theory. In odd dimensions we have the simple formula [51][4]

$$g_{ij}^{(d=odd)} = \frac{16\pi G_N}{\eta d L^{d-1}}\langle T_{ij}\rangle, \tag{5.1.4}$$

with

$$\eta = 1 - 2\left(d(d+1)\lambda_1 + d\lambda_2 + (d-2)(d-3)\lambda_{GB}\right)\frac{\ell^2}{L^2} \tag{5.1.5}$$

In even dimensions the formula is more complicated. For $d = 4$ we discuss the form of the metric in Appendix E

[3]For simplicity we will not include matter fields explicitly in the bulk, but their presence should not alter any of our conclusions.

[4]Even though [51] worked with a flat boundary theory, one can check that this formula remains unchanged when the boundary is curved.

Extremal Surface EWN is a statement about the causal relation between entanglement wedges. To study this, we need to calculate the position of the extremal surface. We parametrize our extremal surface by the coordinate (y^a, z), and the position of the surface is determined by the embedding functions $\bar{X}^\mu(y^a, z)$. The intrinsic metric of the extremal surface is denoted by $\bar{h}_{\alpha\beta}$, where $\alpha = (a, z)$. For convenience we will impose the gauge conditions $\bar{X}^z = z$ and $\bar{h}_{az} = 0$.

The functions $\bar{X}(y^a, z)$ are determined by extremizing the generalized entropy [47, 45] of the entanglement wedge. This generalized entropy consists of geometric terms integrated over the surface as well as bulk entropy terms. We defer a discussion of the bulk entropy terms to §5.3 and write only the geometric terms, which are determined by the bulk action:

$$S_{gen} = \frac{1}{4G_N}\int \sqrt{\bar{h}}\left[1 + 2\lambda_1\ell^2\mathcal{R} + \lambda_2\ell^2\left(\mathcal{R}_{\mu\nu}\mathcal{N}^{\mu\nu} - \frac{1}{2}\mathcal{K}_\mu\mathcal{K}^\mu\right) + 2\lambda_{GB}\ell^2\bar{r}\right]. \tag{5.1.6}$$

We discuss this entropy functional in more detail in Appendix C. The Euler-Lagrange equations for S_{gen} are the equations of motion for $\bar{X}^\mu$. Like the bulk metric, the extremal surface equations can be solved at small-z with a Fefferman–Graham-like expansion:

$$\bar{X}^i(y,z) = X^i_{(0)}(y) + z^2X^i_{(2)}(y) + z^4X^i_{(4)}(y) + \ldots + z^d\log z\, X^i_{(d,log)}(y) + z^dX^i_{(d)}(y) + o(z^d), \tag{5.1.7}$$

As with the metric, the coefficient functions $X^i_{(n)}$ for $n < d$, including the log term, can be solved for in terms of $X^i_{(0)}$ and $g^{(0)}_{ij}$, and again the log term vanishes unless d is even. The state-dependent term $X^i_{(d)}$ contains information about variations of the CFT entropy, as we explain below.

The z-Expansion of EWN By taking the derivative of (5.1.7) with respect to λ, we find the z-expansion of $\delta\bar{X}^i$. We will discuss how to take those derivatives momentarily. But given the z-expansion of $\delta\bar{X}^i$, we can combine this with the z-expansion of $\bar{g}_{ij}$ in (5.1.3) to get the z-expansion of $(\delta\bar{X})^2$:

$$\frac{z^2}{L^2}(\delta\bar{X})^2 = g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)} + z^2\left(2g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(2)} + g^{(2)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)} + X^m_{(2)}\partial_m g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)}\right) + \cdots \tag{5.1.8}$$

EWN implies that $(\delta\bar{X})^2 \geq 0$, and we will spend the next few sections examining this inequality using the expansion (5.1.8). From the general arguments given above, we can get a stronger inequality by considering the vector s^μ and its norm rather than $\delta\bar{X}^\mu$. The construction of s^μ is more involved, but we would similarly construct an equation for s^2 at small z. We defer further discussion of s^μ to §5.1.

Now we return to the question of calculating $\delta\bar{X}^i$. Since all of the $X^i_{(n)}$ for $n < d$ are known explicitly from solving the equation of motion, the λ-derivatives of those terms can be taken and the results expressed in terms of the boundary conditions for the extremal surface. The variation of the state-dependent term, $\delta X^i_{(d)}$, is also determined by the boundary conditions in principle, but in a horribly non-local way. However, we will now show that $X^i_{(d)}$ (and hence $\delta X^i_{(d)}$) can be re-expressed in terms of variations of the CFT entropy.

Variations of the Entropy The CFT entropy S_{CFT} is equal to the generalized entropy S_{gen} of the entanglement wedge in the bulk. To be precise, we need to introduce a cutoff at $z = \epsilon$ and use holographic renormalization to properly define the entropy. Then we can use the calculus of variations to determine variations of the entropy with respect to the boundary conditions at $z = \epsilon$. There will be terms which diverge as $\epsilon \to 0$, as well as a finite term, which is the only one we are interested in at the moment. In odd dimensions, the finite term is given by a simple integral over the entangling surface in the CFT:

$$\delta S_{CFT}\big|_{finite} = \eta d L^{d-1} \int d^{d-2}y \sqrt{h} g_{ij} X^i_{(d)} \delta X^j. \tag{5.1.9}$$

This finite part of S_{CFT} is the renormalized entropy, S_{ren}, in holographic renormalization. Eventually we will want to assure ourselves that our results are scheme-independent. This question was studied in [61], and we will discuss it further in §5.2. For now, the important take-away from (5.1.9) is

$$\frac{1}{\sqrt{h}} \frac{\delta S_{ren}}{\delta X^i(y)} = -\frac{\eta d L^{d-1}}{4 G_N} X^i_{(d,odd)}. \tag{5.1.10}$$

The case of even d is more complicated, and we will cover the $d = 4$ case in Appendix E.

State-Independent Inequalities

The basic EWN inequality is $(\delta \bar{X})^2 \geq 0$. The challenge is to write this in terms of boundary quantities. In this section we will look at the state-independent terms in the expansion of (5.1.8). The boundary conditions at $z = 0$ are given by the CFT entangling surface and background geometry, which we denote by X^i and g_{ij} without a (0) subscript. The variation vector of the entangling surface is the null vector $k^i = \delta X^i$. We can use the formulas of Appendix D to express the other $X^i_{(n)}$ for $n < d$ in terms of X^i and g_{ij}. This allows us to express the state-independent parts of $(\delta \bar{X})^2 \geq 0$ in terms of CFT data. In this subsection we will look at the leading and subleading state-independent parts. These will be sufficient to fully cover the cases $d \leq 5$.

Leading Inequality From (5.1.8), we see that the first term is actually $k_i k^i = 0$. The next term is the one we call the leading term, which is

$$L^{-2} (\delta \bar{X})^2\big|_{z^0} = 2 k_i \delta X^i_{(2)} + g^{(2)}_{ij} k^i k^j + X^m_{(2)} \partial_m g_{ij} k^i k^j. \tag{5.1.11}$$

From (C.10), we easily see that this is equivalent to

$$L^{-2} \, (\delta \bar{X}^i)^2\big|_{z^0} = \frac{1}{(d-2)^2} \theta^2_{(k)} + \frac{1}{d-2} \sigma^2_{(k)}, \tag{5.1.12}$$

where $\sigma_{ab}^{(k)}$ and $\theta_{(k)}$ are the shear and expansion of the null congruence generated by k^i, and are given by the trace and trace-free parts of $k_i K_{ab}^i$, with K_{ab}^i the extrinsic curvature of the entangling surface. This leading inequality is always nonnegative, as required by EWN. Since we are in the small-z limit, the subleading inequality is only relevant when this leading inequality is saturated. So in our analysis below we will focus on the $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$ case, which can always be achieved by choosing the entangling surface appropriately. Note that in $d = 3$ this is the only state-independent term in $(\delta\bar{X})^2$, and furthermore we always have $\sigma_{ab}^{(k)} = 0$ in $d = 3$.

Subleading Inequality The subleading term in $(\delta\bar{X})^2$ is order z^2 in $d \geq 5$, and order $z^2 \log z$ in $d = 4$. These two cases are similar, but it will be easiest to focus first on $d \geq 5$ and then explain what changes in $d = 4$. The terms we are looking for are

$$\begin{aligned} L^{-2}(\delta\bar{X})^2\big|_{z^2} &= 2k_i\delta X_{(4)}^i + 2g_{ij}^{(2)}k^i\delta X_{(2)}^j + g_{ij}\delta X_{(2)}^i\delta X_{(2)}^j + g_{ij}^{(4)}k^ik^j + X_{(4)}^m\partial_m g_{ij}k^ik^j \\ &\quad + 2X_{(2)}^m\partial_m g_{ij}k^i\delta X_{(2)}^j + X_{(2)}^m\partial_m g_{ij}^{(2)}k^ik^j + \frac{1}{2}X_{(2)}^m X_{(2)}^n\partial_m\partial_n g_{ij}k^ik^j. \end{aligned} \tag{5.1.13}$$

This inequality is significantly more complicated than the previous one. The details of its evaluation are left to Appendix D. The result, assuming $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, is

$$\begin{aligned} L^{-2}(\delta\bar{X})^2\big|_{z^2} &= \frac{1}{4(d-2)^2}(D_a\theta_{(k)} + 2R_{ka})^2 \\ &\quad + \frac{1}{(d-2)^2(d-4)}(D_a\theta_{(k)} + R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma_{bc}^{(k)})^2 \\ &\quad + \frac{\kappa}{d-4}\left(C_{kabc}C_k{}^{abc} - 2C_k{}^c{}_{ca}C_k{}^b{}_b{}^a\right). \end{aligned} \tag{5.1.14}$$

where κ is proportional to $\lambda_{GB}\ell^2/L^2$ and is defined in Appendix D. Aside from the Gauss–Bonnet term we have a sum of squares, which is good because EWN requires this to be positive when $\theta_{(k)}$ and $\sigma_{(k)}$ vanish. Since $\kappa \ll 1$, it cannot possibly interfere with positivity unless the other terms were zero. This would require $D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = R_{ka} = 0$ in addition to our other conditions. But, following the arguments of [104], this cannot happen unless the components C_{kabc} of the Weyl tensor also vanish at the point in question. Thus EWN is always satisfied. Also note that the last two terms in middle line of (5.1.14) are each conformally invariant when $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, which we have assumed. This will become important later.

Finally, though we have assumed $d \geq 5$ to arrive at this result, we can use it to derive the expression for $L^{-2}(\delta\bar{X})^2\big|_{z^2\log z}$ in $d = 4$. The rule, explained in Appendix E, is to multiply the RHS by $4 - d$ and then set $d = 4$. This has the effect of killing the conformally non-invariant term, leaving us with

$$L^{-2}(\delta\bar{X})^2\big|_{z^2\log z, d=4} = -\frac{1}{4}(D_a\theta_{(k)} + R_{ka})^2 - \frac{1}{4}(D_a\sigma_{bc}^{(k)})^2. \tag{5.1.15}$$

The Gauss–Bonnet term also disappears because of a special Weyl tensor identity in $d = 4$ [61]. The overall minus sign is required since $\log z < 0$ in the small z limit. In addition, we no longer require that R_{ka} and $D_a\theta_{(k)}$ vanish individually to saturate the inequality: only their sum has to vanish. This still requires that $C_{kabc} = 0$, though.

The Quantum Null Energy Condition

The previous section dealt with the two leading state-independent inequalities that EWN implies. Here we deal with the leading state-*dependent* inequality, which turns out to be the QNEC.

At all orders lower than z^{d-2}, $(\delta\bar{X})^2$ is purely geometric. At order z^{d-2}, however, the CFT energy-momentum tensor enters via the Fefferman–Graham expansion of the metric, and variations of the entropy enter through $X^i_{(d)}$. In odd dimensions the analysis is simple and we will present it here, while in general even dimensions it is quite complicated. Since our state-independent analysis is incomplete for $d > 5$ anyway, we will be content with analyzing only $d = 4$ for the even case. The $d = 4$ calculation is presented in Appendix E. Though is it more involved that the odd-dimensional case, the final result is the same.

Consider first the case where d is odd. Then we have

$$L^{-2}(\delta\bar{X})^2\big|_{z^{d-2}} = g^{(d)}_{ij}k^ik^j + 2k_i\delta X^i_{(d)} + X^m_{(d)}\partial_m g_{ij}k^ik^j = g^{(d)}_{ij}k^ik^j + 2\delta\left(k_i\delta X^i_{(d)}\right). \quad (5.1.16)$$

From (5.1.4) and (5.1.10), we find that

$$L^{-2}(\delta\bar{X})^2\big|_{z^{d-2}} = \frac{16\pi G_N}{\eta d L^{d-1}}\left[\langle T_{kk}\rangle - \delta\left(\frac{k^i}{2\pi\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right)\right]. \quad (5.1.17)$$

The nonnegativity of the term in brackets is equivalent to the QNEC. The case where d is even is more complicated, and we will go over the $d = 4$ case in Appendix E.

The Conformal QNEC

As mentioned in §5.1, we can get a stronger inequality from EWN by considering the norm of the vector s^μ, which is the part of $\delta\bar{X}^\mu$ orthogonal to the extremal surface. Our gauge choice $\bar{X}^z = z$ means that $s^\mu \neq \delta\bar{X}^\mu$, and so we get a nontrivial improvement by considering $s^2 \geq 0$ instead of $(\delta\bar{X})^2 \geq 0$.

We can actually use the results already derived above to compute s^2 with the following trick. We would have had $\delta\bar{X}^\mu = s^\mu$ if the surfaces of constant z were already orthogonal to the extremal surfaces. But we can change our definition of the constant-z surfaces with a coordinate transformation in the bulk to make this the case, apply the above results to $(\delta\bar{X})^2$ in the new coordinate system, and then transform back to the original coordinates. The coordinate transformation we are interested in performing is a PBH transformation [88], since it leaves the metric in Fefferman–Graham form, and so induces a Weyl transformation on the boundary.

So from the field theory point of view, we will just be calculating the consequences of EWN in a different conformal frame, which is fine because we are working with a CFT. With that in mind it is easy to guess the outcome: the best conformal frame to pick is one in which all of the non-conformally-invariant parts of the state-independent terms in $(\delta\bar{X})^2$ are set to zero, and when we transform the state-dependent term in the new frame back to the original frame we get the so-called Conformal QNEC first defined in [96]. This is indeed what happens, as we will now see.

Orthogonality Conditions First, we will examine in detail the conditions necessary for $\delta\bar{X}^\mu = s^\mu$, and their consequences on the inequalities derived above. We must check that

$$\bar{g}_{ij}\partial_\alpha \bar{X}^i \delta\bar{X}^j = 0. \tag{5.1.18}$$

for both $\alpha = z$ and $\alpha = a$. As above, we will expand these conditions in z. When $\alpha = z$, at lowest order in z we find the condition

$$0 = k_i X^i_{(2)}, \tag{5.1.19}$$

which is equivalent to $\theta_{(k)} = 0$. When $\alpha = a$, the lowest-order in z inequality is automatically satisfied because k^i is defined to be orthogonal to the entangling surface on the boundary. But at next-to-lowest order we find the condition

$$\begin{aligned} 0 &= k_i \partial_a X^i_{(2)} + e_{ai}\delta X^i_{(2)} + g^{(2)}_{ij} e^i_a k^j + X^m_{(2)}\partial_m g_{ij} e^i_a k^j && (5.1.20)\\ &= -\frac{1}{2(d-2)}\left[(D_a - 2w_a)\theta_{(k)} + 2R_{ka}\right]. && (5.1.21)\end{aligned}$$

Combined with the $\theta_{(k)} = 0$ condition, this tells us that that $D_a\theta_{(k)} = -2R_{ka}$ is required. When these conditions are satisfied, the state-dependent terms of $(\delta\bar{X})^2$ analyzed above become[5]

$$L^{-2}(\delta\bar{X})^2 = \frac{1}{d-2}\sigma^2_{(k)} + \left[\frac{1}{(d-2)^2(d-4)}(R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma^{(k)}_{bc})^2\right]z^2 + \cdots \tag{5.1.22}$$

Next we will demonstrate that $\theta_{(k)} = 0$ and $D_a\theta_{(k)} = -2R_{ka}$ can be achieved by a Weyl transformation, and then use that fact to write down the $s^2 \geq 0$ inequality that we are after.

Achieving $\delta\bar{X}^\mu = s^\mu$ with a Weyl Transformation Our goal now is to begin with a generic situation in which $\delta\bar{X}^\mu \neq s^\mu$ and use a Weyl transformation to set $\delta\bar{X}^\mu \to s^\mu$. This means finding a new conformal frame with $\hat{g}_{ij} = e^{2\phi(x)}g_{ij}$ such that $\hat{\theta}_{(k)} = 0$ and

[5] We have not included some terms at order z^2 which are proportional to $\sigma^{(k)}_{ab}$ because they never play a role in the EWN inequalities.

$\hat{D}_a\hat{\theta}_{(k)} = -2\hat{R}_{ka}$, which would then imply that $\delta\hat{X}^\mu = s^\mu$ (we omit the bar on $\delta\hat{X}^\mu$ to avoid cluttering the notation, but logically it would be $\delta\hat{\bar{X}}^\mu$).

Computing the transformation properties of the geometric quantities involved is a standard exercise, but there is one extra twist involved here compared to the usual prescription. Ordinarily a vector such as k^i would be invariant under the Weyl transformation. However, for our setup is it is important that k^i generate an affine-parameterized null geodesic. Even though the null geodesic itself is invariant under Weyl transofrmation, k^i will no longer be the correct generator. Instead, we have to use $\hat{k}^i = e^{-2\phi}k^i$. Another way of saying this is that $k_i = \hat{k}_i$ is invariant under the Weyl transformation. With this in mind, we have

$$e^{2\phi}\hat{R}_{ka} = R_{ka} - (d-2)\left[D_a\partial_k\phi - w_a\partial_k\phi - k_j K^j_{ab}\partial^b\phi - \partial_k\phi\partial_a\phi\right], \tag{5.1.23}$$

$$e^{2\phi}\hat{\theta}_{(k)} = \theta_{(k)} + (d-2)\partial_k\phi, \tag{5.1.24}$$

$$e^{2\phi}\hat{D}_a\hat{\theta}_{(k)} = D_a\theta_{(k)} + (d-2)D_a\partial_k\phi - 2\theta_{(k)}\partial_a\phi - 2(d-2)\partial_k\phi\partial_a\phi, \tag{5.1.25}$$

$$\hat{\sigma}^{(k)}_{ab} = \sigma^{(k)}_{ab}, \tag{5.1.26}$$

$$\hat{D}_c\hat{\sigma}^{(k)}_{ab} = D_c\sigma^{(k)}_{ab} - 2\left[\sigma^{(k)}_{c(b}\partial_{a)}\phi + \sigma^{(k)}_{ab}\partial_c\phi - g_{c(a}\sigma^{(k)}_{b)d}\nabla^d\phi\right], \tag{5.1.27}$$

$$\hat{w}_a = w_a - \partial_a\phi. \tag{5.1.28}$$

So we may arrange $\hat{\theta}_{(k)} = 0$ at a given point on the entangling surface by choosing $\partial_k\phi = -\theta_{(k)}/(d-2)$ that that point. Having chosen that, and assuming $\sigma^{(k)}_{ab}$ =0 at the same point, one can check that

$$e^{2\phi}\left(\hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka}\right) = D_a\theta_{(k)} - 2w_a\theta_{(k)} + 2R_{ka} - (d-2)D_a\partial_k\phi \tag{5.1.29}$$

So we can choose $D_a\partial_k\phi$ to make the combination $\hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka}$ vanish. Then in the new frame we have $\delta\hat{X}^\mu = s^\mu$.

The $s^2 \geq 0$ Inequality Based on the discussion above, we were able to find a conformal frame that allows us to compute the s^2. For the state-independent parts we have

$$L^{-2}s^2 = \frac{1}{d-2}\hat{\sigma}^2_{(k)} + \left[\frac{1}{(d-2)^2(d-4)}(\hat{R}_{ka})^2 + \frac{1}{2(d-2)(d-4)}(\hat{D}_a\hat{\sigma}^{(k)}_{bc})^2\right]\hat{z}^2 + \cdots \tag{5.1.30}$$

Here we also have a new bulk coordinate $\hat{z} = ze^\phi$ associated with the bulk PBH transformation. All we have to do now is transform back into the original frame to find s^2. Since $\hat{\theta}_{(k)} = \hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka} = 0$, we actually have that

$$\hat{R}_{ka} = \hat{D}_a\hat{\theta}_{(k)} - \hat{w}_a\hat{\theta}_{(k)} - \hat{R}_{ka}, \tag{5.1.31}$$

which transforms homogeneously under Weyl transformations when $\sigma_{ab}^{(k)} = 0$. Thus, up to an overall scaling factor, we have

$$L^{-2}s^2 = \frac{1}{d-2}\sigma_{(k)}^2 + \left[\frac{1}{(d-2)^2(d-4)}(D_a\theta_{(k)} - w_a\theta_{(k)} - R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma_{bc}^{(k)})^2\right] z^2 + \cdots, \tag{5.1.32}$$

where we have dropped terms of order z^2 which vanish when $\sigma_{ab}^{(k)} = 0$. As predicted, these terms are the conformally invariant contributions to $(\delta\bar{X})^2$.

In order to access the state-dependent part of s^2 we need the terms in (5.1.32) to vanish. Note that in $d = 3$ this always happens. In that case there is no z^2 term, and $\sigma_{ab}^{(k)} = 0$ always. Though our expression is singular in $d = 4$, comparing to (5.1.22) shows that actually the term in brackets above is essentially the same as the $z^2 \log z$ term in $\delta\bar{X}$. We already noted that this term was conformally invariant, so this is expected. The difference now is that we no longer need $\theta_{(k)} = 0$ in order to get to the QNEC in $d = 4$. In $d = 5$ the geometric conditions for the state-independent parts of s^2 to vanish are identical to those for $d = 4$, whereas in the $(\delta\bar{X})^2$ analysis we found that extra conditions were necessary. These were relics of the choice of conformal frame. Finally, for $d > 5$ there will be additional state-independent terms that we have not analyzed, but the results we have will still hold.

Conformal QNEC Now we analyze the state-dependent part of s^2 at order z^{d-2}. When all of the state-independent parts vanish, the state-dependent part is given by the conformal transformation of the QNEC. This is easily computed as follows:

$$L^{-2}\, s^2\big|_{z^{d-2}} = \frac{16\pi G_N}{\eta d L^{d-1}}\left[2\pi\langle\hat{T}_{ij}\rangle k^i k^j - \delta\left(\frac{k^i}{\sqrt{h}}\frac{\delta\hat{S}_{ren}}{\delta X^i(y)}\right) - \frac{d}{2}\theta_{(k)}\left(\frac{k^i}{\sqrt{h}}\frac{\delta\hat{S}_{ren}}{\delta X^i(y)}\right)\right]. \tag{5.1.33}$$

Of course, one would like to replace $\hat{T}_{ij}$ with T_{ij} and $\hat{S}_{ren}$ with S_{ren}. When d is odd this is straightforward, as these quantities are conformally invariant. However, when d is even there are anomalies that will contribute, leading to extra geometric terms in the conformal QNEC [68, 96].

5.2 Connection to Quantum Focusing

The Quantum Focusing Conjecture

We start by reviewing the statement of the QFC [21, 104] before moving on to its connection to EWN and the QNEC. Consider a codimension-two Cauchy-splitting (i.e. entangling) surface Σ and a null vector field k^i normal to Σ. Denote by $\mathcal{N}$ the null surface generated by k^i. The generalized entropy, S_{gen}, associated to Σ is given by

$$S_{gen} = \langle S_{grav} \rangle + S_{ren} \tag{5.2.1}$$

where S_{grav} is a state-independent local integral on Σ and S_{ren} is the renormalized von Neumann entropy of the interior (or exterior of Σ. The terms in S_{grav} are determined by the low-energy effective action of the theory in a well-known way [43]. Even though $\langle S_{grav} \rangle$ and S_{ren} individually depend on the renormalization scheme, that dependence cancels out between them so that S_{gen} is scheme-independent.

The generalized entropy is a functional of the entangling surface Σ, and the QFC is a statement about what happens when we vary the shape of Σ by deforming it within the surface $\mathcal{N}$. Specifically, consider a one-parameter family $\Sigma(\lambda)$ of cuts of $\mathcal{N}$ generated by deforming the original surface using the vector field k^i. Here λ is the affine parameter along the geodesic generated by k^i and $\Sigma(0) \equiv \Sigma$. To be more precise, let y^a denote a set of intrinsic coordinates for Σ, let h_{ab} be the induced metric on Σ, and let $X^i(y, \lambda)$ be the embedding functions for $\Sigma(\lambda)$. With this notation, $k^i = \partial_\lambda X^i$. The change in the generalized entropy is given by

$$\left.\frac{dS_{gen}}{d\lambda}\right|_{\lambda=0} = \int_\Sigma d^{d-2}y \, \frac{\delta S_{gen}}{\delta X^i(y)} \partial_\lambda X^i(y) \equiv \frac{1}{4G_N} \int_\Sigma d^{d-2}y \sqrt{h} \, \Theta[\Sigma, y] \tag{5.2.2}$$

This defines the quantum expansion $\Theta[\Sigma, y]$ in terms of the functional derivative of the generalized entropy:

$$\Theta[\Sigma, y] = 4G_N \frac{k^i(y)}{\sqrt{h}} \frac{\delta S_{gen}}{\delta X^i(y)}. \tag{5.2.3}$$

Note that we have suppressed the dependence of Θ on k^i in the notation, but the dependence is very simple: if $k^i(y) \to f(y) k^i(y)$, then $\Theta[\Sigma, y] \to f(y) \Theta[\Sigma, y]$.

The QFC is simple to state in terms of Θ. It says that Θ is non-increasing along the flow generated by k^i:

$$0 \geq \frac{d\Theta}{d\lambda} = \int_\Sigma d^{d-2}y \, \frac{\delta \Theta[\Sigma, y]}{\delta X^i(y')} k^i(y'). \tag{5.2.4}$$

Before moving on, let us make two remarks about the QFC.

First, the functional derivative $\delta\Theta[\Sigma, y]/\delta X^i(y')$ will contain local terms (i.e. terms proportional to δ-functions or derivatives of δ-functions with support at $y = y'$) as well as non-local terms that have support even when $y \neq y'$. S_{grav}, being a local integral, will only contribute to the local terms of $\delta\Theta[\Sigma, y]/\delta X^i(y')$. The renormalized entropy S_{ren} will contribute both local and non-local terms. The non-local terms can be shown to be nonpositive using strong subadditivity of the entropy [21], while the local terms coming from S_{ren} are in general extremely difficult to compute.

Second, and more importantly for us here, the QFC as written in (5.2.4) does not quite make sense. We have to remember that S_{grav} is really an operator, and its expectation value

$\langle S_{grav}\rangle$ is really the thing that contributes to Θ. In order to be well-defined in the low-energy effective theory of gravity, this expectation value must be smeared over a scale large compared to the cutoff scale of the theory. Thus when we write an inequality like (5.2.4), we are implicitly smearing in y against some profile. The profile we use is arbitrary as long as it is slowly-varying on the cutoff scale. This extra smearing step is necessary to avoid certain violations of (5.2.4), as we will see below [104].

QNEC from QFC

In this section we will explicitly evaluate the QFC inequality, (5.2.4), and derive the QNEC in curved space from it as a nongravitational limit. We consider theories with a gravitational action of the form

$$I_{grav} = \frac{1}{16\pi G_N}\int \sqrt{g}\left(R + \ell^2\lambda_1 R^2 + \ell^2\lambda_2 R_{ij}R^{ij} + \ell^2\lambda_{GB}\mathcal{L}_{GB}\right) \tag{5.2.5}$$

where $\mathcal{L}_{GB} = R_{ijmn}^2 - 4R_{ij}^2 + R^2$ is the Gauss-Bonnet Lagrangian. Here ℓ is the cutoff length scale of the effective field theory, and the dimensionless couplings λ_1, λ_2, and λ_{GB} are assumed to be renormalized.

The generalized entropy functional for these theories can be computed using standard replica methods [43] and takes the form

$$S_{gen} = \frac{A[\Sigma]}{4G_N} + \frac{\ell^2}{4G_N}\int_\Sigma \sqrt{h}\left[2\lambda_1 R + \lambda_2\left(R_{ij}N^{ij} - \frac{1}{2}K_iK^i\right) + 2\lambda_{GB}r\right] + S_{ren}. \tag{5.2.6}$$

Here $A[\Sigma]$ is the area of the entangling surface, N^{ij} is the projector onto the normal space of Σ, K^i is the trace of the extrinsic curvature of Σ, and r is the intrinsic Ricci scalar of Σ.

We can easily compute Θ by taking a functional derivative of (5.2.6), taking care to integrate by parts so that the result is proportional to $k^i(y)$ and not derivatives of $k^i(y)$. One finds

$$\Theta = \theta_{(k)} + \ell^2\Bigg[2\lambda_1(\theta_{(k)}R + \nabla_k R) + \lambda_2\Big((D_a - w_a)^2\theta_{(k)} + K_iK^{iab}K^k_{ab} \tag{5.2.7}$$

$$+ \theta_{(k)}R_{klkl} + \nabla_k R - 2\nabla_l R_{kk} + \theta_{(k)}R_{kl} - \theta_{(l)}R_{kk} + 2K^{kab}R_{ab}\Big)$$

$$- 4\lambda_{GB}\left(r^{ab}K^k_{ab} - \frac{1}{2}r\theta_{(k)}\right)\Bigg] + 4G_N\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i} \tag{5.2.8}$$

Now we must compute the λ-derivative of Θ. When we do this, the leading term comes from the derivative of $\theta_{(k)}$, which by Raychaudhuri's equation contains the terms $\theta_{(k)}^2$ and $\sigma_{(k)}^2$. Since we are ultimately interested in deriving the QNEC as the non-gravitational limit of the QFC, we need to set $\theta_{(k)} = \sigma^{(k)}_{ab} = 0$ so that the nongravitational limit is not dominated by those terms. So for the rest of this section we will set $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$ at the point of evaluation (but not globally!). Then we find

$$
\begin{aligned}
\frac{d\Theta}{d\lambda} &= -R_{kk} + 2\lambda_1\ell^2\left(\nabla_k^2 R - RR_{kk}\right)\\
&+ \lambda_2\ell^2\Big[2D_a(w^a R_{kk}) + \nabla_k^2 R - D_a D^a R_{kk} - \frac{d}{d-2}(D_a\theta_{(k)})^2 - 2R_{kb}D^b\theta_{(k)} - 2(D_a\sigma_{bc})^2\\
&- 2\nabla_k\nabla_l R_{kk} - 2R_{kakb}R^{ab} - \theta_{(l)}\nabla_k R_{kk}\Big] - 2\lambda_{GB}\ell^2\Bigg[\frac{d(d-3)(d-4)}{(d-1)(d-2)^2}RR_{kk}\\
&- 4\frac{(d-4)(d-3)}{(d-2)^2}R_{kk}R_{kl} - \frac{2(d-4)}{d-2}C_{klkl}R_{kk} - \frac{2(d-4)}{d-2}R^{ab}C_{akbk} + 4C^{kalb}C_{kakb}\Bigg]\\
&+ 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right)
\end{aligned}
\tag{5.2.9}
$$

This expression is quite complicated, but it simplifies dramatically if we make use of the equation of motion coming from (5.2.5) plus the action of the matter sector. Then we have $R_{kk} = 8\pi G T_{kk} - H_{kk}$ where [71]

$$
\begin{aligned}
H_{kk} &= 2\lambda_1\left(RR_{kk} - \nabla_k^2 R\right) + \lambda_2\Big(2R_{kikj}R^{ij} - \nabla_k^2 R + 2\nabla_k\nabla_l R_{kk} - 2R_{klki}R_k^i\\
&+D_cD^cR_{kk} - 2D_c(w^cR_{kk}) - 2(D_b\theta_{(k)} + R_{bmkj}P^{mj})R_k^b + \theta_{(l)}\nabla_k R_{kk}\Big)\\
&+ 2\lambda_{GB}\left(\frac{d(d-3)(d-4)}{(d-1)(d-2)^2}RR_{kk} - 4\frac{(d-4)(d-3)}{(d-2)^2}R_{kk}R_{kl} - 2\frac{d-4}{d-2}R^{ij}C_{kikj} + C_{kijm}C_k{}^{ijm}\right)
\end{aligned}
\tag{5.2.10}
$$

For the Gauss-Bonnet term we have used the standard decomposition of the Riemann tensor in terms of the Weyl and Ricci tensors. Using similar methods to those in Appendix D, we have also exchanged $k^ik^j\Box R_{ij}$ in the R_{ij}^2 equation of motion for surface quantities and ambient curvatures.

After using the equation of motion we have the relatively simple formula

$$
\begin{aligned}
\frac{d\Theta}{d\lambda} &= -\lambda_2\ell^2\left(\frac{d}{d-2}(D_a\theta_{(k)})^2 + 4R_k^bD_b\theta_{(k)} + 2R_{bk}R_k^b + 2(D_a\sigma_{bc}^{(k)})^2\right)\\
&+ 2\lambda_{GB}\ell^2\left(C_{kabc}C_k{}^{abc} - 2C_{kba}{}^bC_{kc}{}^{ac}\right) + 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - 8\pi G_N\langle T_{kk}\rangle
\end{aligned}
\tag{5.2.11}
$$

The Gauss-Bonnet term agrees with the expression derived in [61]. However unlike [61] we have not made any perturbative assumptions about the background curvature.

At first glance it seems like (5.2.11) does not have definite sign, even in the non-gravitational limit, due to the geometric terms proportional to λ_2 and λ_{GB}. The difficulty posed by the Gauss-Bonnet term, in particular, was first pointed out in [62]. However, this is where we have to remember the smearing prescription mentioned in §5.2. We must integrate (5.2.11) over a region of size larger than ℓ before testing its nonpositivity. The crucial point, used in [104], is that we must also remember to integrate the terms $\theta_{(k)}^2$ and $\sigma_{(k)}^2$ that we dropped earlier over the same region. When we integrate $\theta_{(k)}^2$ over a region of size ℓ centered at a point where $\theta_{(k)} = 0$, the result is $\xi\ell^2(D_a\theta_{(k)})^2 + o(\ell^2)$, where $\xi \gtrsim 10$ is a parameter associated with the smearing profile. A similar result holds for $\sigma_{ab}^{(k)}$. Thus we arrive at

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -\frac{\xi}{d-2}\ell^2(D_a\theta_{(k)})^2 - \xi\ell^2(D_a\sigma_{bc}^{(k)})^2 \\ &- \lambda_2\ell^2\left(\frac{d}{d-2}(D_a\theta_{(k)})^2 + 4R_k^bD_b\theta_{(k)} + 2R_{bk}R_k^b + 2(D_a\sigma_{bc}^{(k)})^2\right) \\ &+ 2\lambda_{GB}\ell^2\left(C_{kabc}C_k{}^{abc} - 2C_{kba}{}^bC_{kc}{}^{ac}\right) \\ &+ 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - 8\pi G_N\langle T_{kk}\rangle + o(\ell^2)\end{aligned} \tag{5.2.12}$$

Since the size of ξ is determined by the validity of the effective field theory, by construction the terms proportional to ξ in (5.2.12) dominate over the others. Thus in order to take the non-gravitational limit, we must eliminate these smeared terms.

Clearly we need to be able to choose a surface such that $D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = 0$. Then smearing $\theta_{(k)}^2$ and $\sigma_{(k)}^2$ would only produce terms of order ℓ^4 (terms of that order would also show up from smearing the operators proportional to λ_2 and λ_{GB}). As explained in [104], this is only possible given certain conditions on the background spacetime at the point of evaluation. We must have

$$C_{kabc} = \frac{1}{d-2}h_{ab}R_{kc} - \frac{1}{d-2}h_{ac}R_{kb}. \tag{5.2.13}$$

This can be seen by using the Codazzi equation for Σ. Imposing this condition, which allows us to set $D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = 0$, we then have.

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -2\ell^2\left(\lambda_2 + 2\frac{(d-3)(d-4)}{(d-2)^2}\lambda_{GB}\right)R_{bk}R_k^b \\ &+ 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - 8\pi G_N\langle T_{kk}\rangle + o(\ell^3).\end{aligned} \tag{5.2.14}$$

This is the quantity which must be negative according to the QFC. In deriving it, we had to assume that $\theta_{(k)} = \sigma^{(k)} = D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = 0$.

We make two observations about (5.2.14). First, if we assume that $R_{ka} = 0$ as an additional assumption and take $\ell \to 0$, then we arrive at the QNEC as long as $G_N > o(\ell^3)$. This is the case when ℓ scales with the Planck length and $d \leq 5$. These conditions are similar to the ones we found previously from EWN, and below in §5.2 we will discuss that in more detail.

The second observation has to do with the lingering possibility of a violation of the QFC due to the terms involving the couplings. In order to have a violation, one would need the linear combination

$$\lambda_2 + 2\frac{(d-3)(d-4)}{(d-2)^2}\lambda_{GB} \tag{5.2.15}$$

to be negative. Then if one could find a situation where the first line of (5.2.14) dominated over the second, there would be a violation. It would be interesting to interpret this as a bound on the above linear combination of couplings coming from the QFC, but it is difficult to find a situation where the first line of (5.2.14) dominates. The only way for R_{ka} to be large compared to the cutoff scale is if T_{ka} is nonzero, in which case we would have $R_{ka} \sim G_N T_{ka}$. Then in order for the first line of (5.2.14) to dominate we would need

$$G_N \ell^2 T_{ka} T_k^a \gg T_{kk}. \tag{5.2.16}$$

As an example, for a scalar field Φ this condition would say

$$G_N \ell^2 (\partial_a \Phi)^2 \gg 1. \tag{5.2.17}$$

This is not achievable within effective field theory, as it would require the field to have super-Planckian gradients. We leave a detailed and complete discussion of this issue to future work.

Scheme-Independence of the QNEC

We take a brief interlude to discuss the issue of the scheme-dependence of the QNEC, which will be important in the following section. It was shown in [61], under some slightly stronger assumptions than the ones we have been using, that the QNEC is scheme-independent under the same conditions where we expect it to hold true. Here we will present our own proof of this fact, which actually follows from the manipulations we performed above involving the QFC.

In this section we will take the point of view of field theory on curved spacetime without dynamical gravity. Then each of the terms in I_{grav}, defined above in (5.2.5), are completely arbitrary, non-dynamical terms we can add to the Lagrangian at will.[6] Dialing the values of those various couplings corresponds to a choice of *scheme*, as even though those couplings are non-dynamical they will still contribute to the definitions of quantities like the renormalized energy-momentum tensor and the renormalized entropy (as defined through the replica trick). The QNEC is scheme-independent if it is insensitive to the values of these couplings.

To show the scheme-independence of the QNEC, we will begin with the statement that S_{gen} is scheme-independent. We remarked on this above, when our context was a theory with dynamical gravity. But the scheme-independence of S_{gen} does not require use of the equations of motion, so it is valid even in a non-gravitational theory on a fixed background.

[6] We should really be working at the level of the quantum effective action, or generating functional, for correlation functions of T_{ij} [62]. The geometrical part has the same form as the classical action I_{grav} and so does not alter this discussion.

In fact, only once in the above discussion did we make use of the gravitational equations of motion, and that was in deriving (5.2.11). Following the same steps up to that point, but without imposing the gravitational equations of motion, we find instead

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -\lambda_2\ell^2\left(\frac{d}{d-2}(D_a\theta_{(k)})^2 + 4R_k^bD_b\theta_{(k)} + 2R_{bk}R_k^b + 2(D_a\sigma_{bc})^2\right)\\ &\quad + 2\lambda_{GB}\ell^2\left(C_{kabc}C_k{}^{abc} - 2C_{kba}{}^bC_{kc}{}^{ac}\right) + 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - k_ik_j\frac{16\pi G_N}{\sqrt{g}}\frac{\delta I_{grav}}{\delta g_{ij}}.\end{aligned} \tag{5.2.18}$$

Since the theory is not gravitational, we would not claim that this quantity has a sign. However, it is still scheme-independent.

To proceed, we will impose all of the additional conditions that are necessary to prove the QNEC. That is, we impose $D_b\theta_{(k)} = R_k^b = D_a\sigma_{bc} = 0$, as well as $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, which in turn requires $C_{kabc} = 0$. Under these conditions, we learn that the combination

$$\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - k_ik_j\frac{4\pi}{\sqrt{g}}\frac{\delta I_{grav}}{\delta g_{ij}} \tag{5.2.19}$$

is scheme-independent. The second term here is one of the contributions to the renormalized $2\pi\langle T_{kk}\rangle$ in the non-gravitational setup, the other contribution being $k_ik_j\frac{4\pi}{\sqrt{g}}\frac{\delta I_{matter}}{\delta g_{ij}}$. But I_{matter} is already scheme-independent in the sense we are discussing, in that it is independent of the parameters appearing in I_{grav}. So adding that to the terms we have above, we learn that

$$\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right) - 2\pi\langle T_{kk}\rangle \tag{5.2.20}$$

is scheme-independent. This is what we wanted to show.

QFC vs EWN

As we have discussed above, by taking the non-gravitational limit of (5.2.14) under the assumptions $D_b\theta_{(k)} = R_k^b = D_a\sigma_{bc} = \theta_{(k)} = \sigma_{ab}^{(k)} = 0$ we find the QNEC as a consequence of the QFC (at least for $d \leq 5$). And under the same set of geometric assumptions, we found the QNEC as a consequence of EWN in (5.1.17). The discussion of the previous section demonstrates that these assumptions also guarantee that the QNEC is scheme-independent. So even though these two QNEC inequalities were derived in different ways, we know that at the end of the day they are the same QNEC. It is natural to ask if there is a further relationship between EWN and the QFC, beyond the fact that they give the same QNEC. We will begin to investigate that question in this section.

The natural thing to ask about is the state-independent terms in the QFC and in $(\delta\bar{X})^2$. We begin by writing down all of the terms of $(\delta\bar{X})^2$ in odd dimensions that we have computed:

$$\begin{aligned}(d-2)L^{-2}(\delta\bar{X}^i)^2 = &\frac{1}{(d-2)}\theta_{(k)}^2 + \sigma_{(k)}^2 \\ &+ z^2\frac{1}{4(d-2)}(D_a\theta_{(k)} + 2R_{ka})^2 \\ &+ z^2\frac{1}{(d-2)(d-4)}(D_a\theta_{(k)} + R_{ka})^2 + z^2\frac{1}{2(d-4)}(D_a\sigma_{bc}^{(k)})^2 \\ &+ z^2\frac{\kappa}{d-4}\left(C_{kabc}C_k^{\ abc} - 2C_{k\ ca}^{\ c}C_{k\ b}^{\ b\ a}\right) \\ &+ \cdots + z^{d-2}\frac{16\pi(d-2)G_N}{\eta d L^{d-1}}\left[\langle T_{kk}\rangle - \delta\left(\frac{k^i}{2\pi\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right)\right]. \end{aligned} \tag{5.2.21}$$

The first line looks like $-\dot{\theta}$, which would be the leading term in $d\Theta/d\lambda$, except it is missing an R_{kk}. Of course, we eventually got rid of the R_{kk} in the QFC by using the equations of motion. Suppose we set $\theta_{(k)} = 0$ and $\sigma_{ab}^{(k)} = 0$ to eliminate those terms, as we did with the QFC. Then we can write $(\delta\bar{X})^2$ suggestively as

$$\begin{aligned}(d-2)L^{-2}(\delta\bar{X}^i)^2 = &z^2\tilde{\lambda}_2\Big(\frac{d}{(d-2)}(D_a\theta_k)^2 + 4R_k^a D_a\theta + \frac{4(d-3)}{(d-2)}R_{ka}R_k^a + 2(D_a\sigma_{bc}^{(k)})^2\Big) \\ &- 2z^2\tilde{\lambda}_{GB}\left(C_{kabc}C_k^{\ abc} - 2C_{k\ ca}^{\ c}C_{k\ b}^{\ b\ a}\right) \\ &+ \cdots + 8\pi\tilde{G}_N\langle T_{kk}\rangle - 4\tilde{G}_N\delta\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{ren}}{\delta X^i}\right). \end{aligned} \tag{5.2.22}$$

where

$$\tilde{G}_N = G_N\frac{2(d-2)z^{d-2}}{\eta d L^{d-1}}, \tag{5.2.23}$$

$$\tilde{\lambda}_2 = \frac{1}{4(d-4)}, \tag{5.2.24}$$

$$\tilde{\lambda}_{GB} = -\frac{\kappa}{2(d-4)}. \tag{5.2.25}$$

Written this way, it almost seems like $(d-2)L^{-2}(\delta\bar{X}^i)^2 \sim -d\Theta/d\lambda$ in some kind of model gravitational theory. One discrepancy is in the coefficient of the $R_{ka}R^{ka}$ term, unless $d = 4$. It is also intriguing that the effective coefficients $\tilde{G}_N$, $\tilde{\lambda}_2$, and $\tilde{\lambda}_{GB}$ are close to, but not exactly the same as, the effective braneworld induced gravity coefficients found in [116]. This is clearly something that deserves further study.

5.3 Discussion

We have displayed a strong similarity between the state-independent inequalities in the QFC and the state-independent inequalities from EWN. We now discuss several possible future directions and open questions that follow naturally from these results.

Bulk Entropy Contributions

We ignored the bulk entropy S_{bulk} in this work, but we know that it produces a contribution to CFT entropy [50] and plays a role in the position of the extremal surface [47, 45]. The bulk entropy contributions to the entropy are subleading in N^2 and do not interfere with the gravitational terms in the entropy. We could include the bulk entropy as a source term in the equations determining $\bar{X}$, which could lead to extra contributions to the $X_{(n)}$ coefficients. However, it does not seem possible for the bulk entropy to have an effect on the state-independent parts of the extremal surface, namely on $X_{(n)}$ for $n < d$, which means the bulk entropy would not affect the conditions we derived for when the QNEC should hold.

Another logical possibility is that the bulk entropy term could affect the statement of the QNEC itself, meaning that the schematic form $T_{kk} - S''$ would be altered. This would be problematic, especially given that the QFC always produces a QNEC of that same form. It was argued in [3] that this does not happen, and that argument holds here as well.

Smearing of EWN

We were careful to include a smearing prescription for defining the QFC, and it was an important ingredient in the analysis of §5.2. But what about smearing of EWN? Of course, the answer is that we *should* smear EWN appropriately, but as we will see now it would not make a difference to our analysis.

The issue is that the bulk theory is a low-energy effective theory of gravity with a cutoff scale ℓ, and the quantities that we use to probe EWN, like $(\delta\bar{X})^2$, are operators in that theory. As such, these operators need to be smeared over a region of proper size ℓ on the extremal surface. Of course, due to the warp factor, such a region has coordinate size $z\ell/L$. We can ask what effect such a smearing would have on the inequality $(\delta\bar{X})^2$.

When we performed our QNEC derivation, we assumed that $\theta_{(k)} = 0$ at the point of evaluation, so that the $\theta_{(k)}^2$ term in $(\delta\bar{X})^2\big|_{z^0}$ would not contribute. However, after smearing this term would contribute a term of the form $\ell^2(D_a\theta_{(k)})^2/L^2$ to $(\delta\bar{X})^2\big|_{z^2}$. But we already had such a term at this order, so all this does is shift the coefficient. Furthermore, the coefficient is shifted only by an amount of order ℓ^2/L^2. If the cutoff ℓ is of order the Planck scale, then this is suppressed in powers of N^2. In other words, this effect is negligible for the analysis. A similar statement applies for $\sigma_{ab}^{(k)}$. So in summary, EWN should be smeared, but the analysis we performed was insensitive to it.

Future Work

There are a number of topics that merit investigation in future work. We will touch on a few of them to finish our discussion.

Relevant Deformations Perhaps the first natural extension of our work is to include relevant deformations in the EWN calculation. There are a few reasons why this is interesting. First, one would like to test the continued correspondence between the QFC and EWN when it comes to the QNEC. The QFC arguments do not care whether relevant deformations are turned on, so one would expect that the same is true in EWN. This is indeed the case when the boundary theory is formulated on flat space [96], and one would expect similar results to hold when the boundary is curved.

Another reason to add in relevant deformations is to test the status of the Conformal QNEC when the theory is not a CFT. To be more precise, the $(\delta\bar{X})^2$ and s^2 calculations we performed differed by a Weyl transformation on the boundary, and since our boundary theory was a CFT this was a natural thing to do. When the boundary theory is not a CFT, what is the relationship between $(\delta\bar{X})^2$ and s^2? One possibility, perhaps the most likely one, is that they simply reduce to the same inequality, and the Conformal QNEC no longer holds.

Finally, and more speculatively, having a relevant deformation turned on when the background is curved allows for interesting state-independent inequalities from EWN. We saw that for a CFT the state-independent terms in both $(\delta\bar{X})^2$ and s^2 were trivially positive. Perhaps when a relevant deformation is turned on more nontrivial results uncover themselves, such as the possibility of a c-theorem hiding inside of EWN. We are encouraged by the similarity of inequalities used in recent proofs of the c-theorems to inequalities obtained from EWN [38].

Higher Dimensions Another pressing issue is extending our results to $d = 6$ and beyond. This is an algebraically daunting task using the methods we have used for $d \leq 5$. Considering the ultimate simplicity of our final expressions, especially compared to the intermediate steps in the calculations, it is likely that there are better ways of formulating and performing the analyses we performed here. It is hard to imagine performing the full $d = 6$ analysis without such a simplification.

Further Connections Between EWN and QFC Despite the issues outlined in §5.2, we are still intrigued by the similarities between EWN and the QFC. It is extremely natural to couple the boundary theory in AdS/CFT to gravity using a braneworld setup [123, 134, 70, 116]. Upon doing this, one can formulate the QFC on the braneworld. However, at the same time near-boundary EWN becomes lost, or at least changes form: extremal surfaces anchored to a brane will in general not be orthogonal to the brane, and in that case a null deformation on the brane will induce a timelike deformation of the extremal surface in the vicinity of the brane. Of course, one has to be careful to take into account the uncertainty in the position of the brane since we are dealing with expectation values of operators, which complicates things. We hope that such an analysis could serve to unify the QFC with EWN, or at least illustrate their relationship with each other.

Conformal QNEC from QFC While we emphasized the apparent similarity between the EWN-derived inequality $(\delta \bar{X})^2 \geq 0$ and the QFC, the stronger EWN inequality $s^2 \geq 0$ is nowhere to be found in the QFC discussion. It would be interesting to see if there is a direct QFC calculation that yields the Conformal QNEC (rather than first deriving the ordinary QNEC and then performing a Weyl transformation). In particular, the Conformal QNEC applies even in cases where $\theta_{(k)}$ is nonzero, while in those cases the QFC is dominated by classical effects. Perhaps there is a useful change of variables that one can do in the semiclassical gravity when the matter sector is a CFT which makes the Conformal QNEC manifest from the QFC point of view. This is worth exploring.

Acknowledgements

It is a pleasure to thank R. Bousso, J. Koeller, and A. Wall for discussions. Our work is supported in part by the Berkeley Center for Theoretical Physics, by the National Science Foundation (award numbers 1521446, and 1316783), by FQXi, and by the US Department of Energy under contract DE-AC02-05CH11231.

Chapter 6

Energy Density from Second Shape Variations of the von Neumann Entropy

6.1 Introduction

The connection between quantum information and energy has been an emerging theme of recent progress in quantum field theory. Causality combined with universal inequalities like positivity and monotonicity of relative entropy can be used to derive many interesting energy-entropy bounds. Examples include the Bekenstein bound [36], the quantum Bousso bound [25, 23], the Averaged Null Energy Condition (ANEC) [52, 76], and the Quantum Null Energy Condition (QNEC) [27, 9]. Here we strengthen the energy-entropy connection, moving from bounds to equalities.

The key insight of the QNEC, which we will exploit, is that one should look at variations of the entropy S of a region as the region is deformed. Consider the entropy as a functional of the entangling surface embedding functions X^μ. Then one can compute the functional derivative $\delta^2 S/\delta X^\mu(y)\delta X^\nu(y')$ which encodes how the entropy depends on the shape of the region. In general, this second variation will contain contact, or "diagonal," terms, proportional to δ-functions and derivatives of δ-functions, as well as "off-diagonal" terms. Our interest here is in the δ-function contact term, and we introduce $S''_{\mu\nu}$ as the coefficient of the δ-function:

$$\frac{\delta^2 S}{\delta X^\mu(y)\delta X^\nu(y')} = S''_{\mu\nu}(y)\delta^{(d-2)}(y-y') + \cdots \tag{6.1.1}$$

Null Variations First consider the null-null component of the second variation, $S''_{vv}(y)$, where v is a null coordinate in a direction orthogonal to the entangling surface at the point y.[1] Suppose the entangling surface is locally restricted to lie in the null plane orthogonal to

[1]We are restricting attention to field theories in Minkowski space throughout the main text.

v near the point y. With this setup the QNEC applies, which says $S''_{vv} \leq 2\pi\langle T_{vv}\rangle$. Our main conjecture in this context is that the QNEC inequality is always saturated:

$$S''_{vv} = 2\pi\langle T_{vv}\rangle. \tag{6.1.2}$$

We believe this holds for all relativistic quantum field theories with an interacting UV fixed point in $d > 2$ dimensions. For a CFT this fully specifies the stress tensor in terms of entropy variations: by considering (6.1.2) for all entangling surfaces passing through a point, $\langle T_{\mu\nu}\rangle$ is completely determined up to a trace term, which would vanish for a CFT. This is the sense in which energy comes from entanglement.

Our primary evidence for (6.1.2) is holographic, as explained below. But if we restrict attention to quantities that can be built out of local expectation values of operators and the local surface geometry there is no other possibility for S''_{vv}. A significant constraint comes from considering the vacuum modular Hamiltonian, K, which is defined by

$$S(\sigma+\delta\sigma) - S(\sigma) = \mathrm{Tr}\,(K\delta\sigma) + O\big(\delta\sigma^2\big)\,, \tag{6.1.3}$$

where σ is the vacuum state reduced to the region under consideration and $\delta\sigma$ is an arbitrary perturbation of the state. If we had a general formula for S in terms of expectation values of operators, we would be able to read off the modular Hamiltonian from the terms in that formula linear in expectation values.[2] For a region bounded by an entangling surface restricted to a null plane the modular Hamiltonian has a known formula in terms of the stress tensor [39], and in particular we have

$$K''_{vv} = 2\pi T_{vv}. \tag{6.1.4}$$

That is why $\langle T_{vv}\rangle$ is the only possible linear term we could have had in (6.1.2).

A nonlinear contribution to S''_{vv}, such as a product of expectation values, is restricted by dimensional analysis and unitarity bounds: the only possibility is if the theory contains a free scalar, in which case we can act with two derivatives on a product of two scalar expectation values to get a viable contribution to S''_{vv}. We will say more about free theories in Appendix G, where we will see that this possibility is realized by a term $\sim \langle\partial_v\phi\rangle^2$, which is why we limit ourselves to interacting theories in the main text. The substance of (6.1.2), then, is the statement that there are no non-local contributions to S''_{vv}.

Relative Entropy There is a natural interpretation of (6.1.2) in terms of relative entropy. The relative entropy of a state ρ and a reference state σ—for us, the vacuum—is a measure of the distinguishability of the two states. We will denote the relative entropy of ρ and the vacuum by $S_{\mathrm{rel}}(\rho)$. By definition, the relative entropy is

$$S_{\mathrm{rel}}(\rho) = \Delta\langle K\rangle - \Delta S, \tag{6.1.5}$$

[2] For simplicity of the discussion we set all vacuum expectation values to zero.

where $\Delta\langle K\rangle$ and ΔS denote the vacuum-subtracted modular energy and vacuum-subtracted entropy, respectively. A consequence of (6.1.2) is that $\Delta S''_{vv} = \Delta\langle K''_{vv}\rangle$, so we can say that

$$S''_{\mathrm{rel},vv} = 0. \tag{6.1.6}$$

This equation is implied by (6.1.2) but is weaker, since it does not require us to know what the modular Hamiltonian actually is. The extra information of (6.1.2) is the expression (6.1.4) for the second variation of the modular Hamiltonian. It can be useful to formulate our results in terms relative entropy instead of entropy itself because relative entropy is generally finite, at least for nice regions.

Non-Null Deformations Now let us move beyond the null case. As explained in [5, 61] and below in Section 6.2, (6.1.2) is a well-defined, finite equation in field theory. Local stationarity conditions on the entangling surface are enough to eliminate state-independent geometric divergences in the entropy, and the remaining state-dependent divergences cancel between the entropy and stress tensor. In the general case, eliminating these divergences is more difficult. State-independent divergences can be dealt with by considering the vacuum-subtracted entropy ΔS rather than just S. State-dependent divergences associated with low-lying operators in the theory are more problematic. To eliminate these divergences, it is enough to restrict our attention to CFTs in states where operators of dimension $\Delta < d/2$ have vanishing expectation values near the entangling surface. Then we find

$$\Delta S''_{\mu\nu} = 2\pi\left(n_\mu^\rho n_\nu^\sigma \langle T_{\rho\sigma}\rangle + \frac{d^2-3d-2}{2(d+1)(d-2)} n_{\mu\nu} h^{ab}\langle T_{ab}\rangle\right), \tag{6.1.7}$$

where $n_{\mu\nu}$ is the normal projector to the entangling surface and h_{ab} is the intrinsic metric on the entangling surface. Note that (6.1.7) implies that $S''_{\mathrm{rel},\mu\nu} = 0$.

We view (6.1.2) and (6.1.7) as deep truths about interacting quantum field theories, worthy of further study. At present, our evidence for these conjectures comes from holography. We will calculate $S''_{\mu\nu}$ directly and prove that (6.1.2) and (6.1.7) hold precisely at leading order in large-N for all bulk states. We will also argue that subleading corrections in $1/N$ do not alter these conclusions. While this does not amount to a full proof, it is enough evidence for us to posit that (6.1.2) and (6.1.7) are true universally.

Outline In Section 6.2 we review some of the basic concepts of entropy, relative entropy, and the holographic setup that will be relevant for our calculation. In Section 6.3 we prove (6.1.2) for situations where it is sufficient to consider linear perturbations of the bulk geometry. This includes any state where gravitational backreaction in the bulk is small. In Section 6.4 we extend this proof to any bulk state. The idea is that S''_{vv} is related to near-boundary physics in the bulk, and for any state the near-boundary geometry is approximately vacuum. So the proof reduces to the linear case. In Section 6.5 we move away from null deformations to prove (6.1.7) using the same techniques. We conclude in Section 6.6 with a discussion of extensions and implications of our work. Several appendices are included discussing closely related topics.

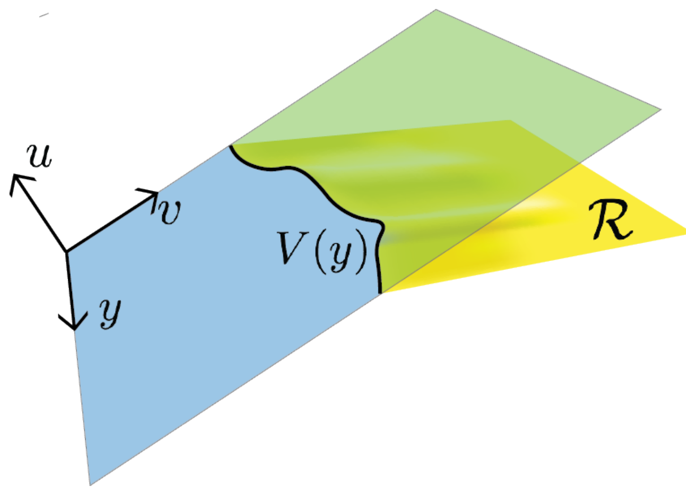

Figure 6.1: Most of our work concerns the variations of entanglement entropy for the yellow region $\mathcal{R}$ whose boundary $\partial\mathcal{R}$ lies on the null plane $u = 0$. The entangling surface is specified by the function $V(y)$.

6.2 Setup and Conventions

In this section we will make some general remarks about the known relations between entropy and energy, and the implications of our conjecture.

The Field Theory Setup

Let $u = (t - x)/\sqrt{2}$ and $v = (t + x)/\sqrt{2}$ be null coordinates, and let y denote the other $d - 2$ spatial coordinates. For now, and for most of the rest of the paper, we will take the boundary of our region $\partial\mathcal{R}$ to be a section of the null plane $u = 0$. This boundary is specified by the equation $v = V(y)$. We take the region $\mathcal{R}$ to be a surface lying witin the "right quadrant," having $u < 0$ and $v > V(y)$ (marked in yellow in Fig 6.1). A one-parameter family of functions $V_\lambda(y)$ specifies a one-parameter family of regions $\mathcal{R}(\lambda)$. We always take the one-parameter family to be of the form $V_\lambda(y) = V_0(y) + \lambda\dot{V}(y)$ with $\dot{V} \geq 0$, so that λ plays the roll of an affine parameter along a future-directed null geodesic located at position y.

Given any global state of the theory, we can compute the von Neumann entropy S of the region $\mathcal{R}$. Keeping the state fixed, the entropy becomes a functional of the boundary of the region, $S = S[V(y)]$. When we have a one-parameter family of regions, then we can write $S(\lambda) = S[V_\lambda(y)]$. Throughout the rest of this work we will be interested in the derivatives of S with respect to λ, as well as the functional derivatives of S with respect to $V(y)$. These are related by the chain rule:

$$\frac{dS}{d\lambda} = \int d^{d-2}y \; \frac{\delta S}{\delta V(y)} \dot{V}(y), \tag{6.2.1}$$

$$\frac{d^2 S}{d\lambda^2} = \int d^{d-2}y d^{d-2}y' \; \frac{\delta^2 S}{\delta V(y) \delta V(y')} \dot{V}(y) \dot{V}(y'). \tag{6.2.2}$$

We can parametrize the second functional derivative as follows:

$$\frac{\delta^2 S}{\delta V(y) \delta V(y')} = S''_{vv}(y) \delta^{(d-2)}(y - y') + \frac{\delta^2 S^{\text{od}}}{\delta V(y) \delta V(y')}. \tag{6.2.3}$$

We have extracted a δ-function terms explicitly, which we sometimes refer to as the "diagonal" part, and the remainder carries the label "od" for "off-diagonal." Note that the off-diagonal part of the variation does not have to vanish at $y = y'$. The quantity S''_{vv} is the same as S'' in [26, 97, 21].

In addition to the entropy of the region $\mathcal{R}$, we can define the vacuum-subtracted modular energy, $\Delta\langle K \rangle$, and relative entropy with respect to the vacuum, S_{rel}, associated to the region $\mathcal{R}$. The modular energy is given by the boost energy along each generator of the null plane [39]:

$$\Delta\langle K \rangle = 2\pi \int d^{d-2}y \int_{V(y)}^{\infty} dv \; (v - V(y)) \langle T_{vv} \rangle. \tag{6.2.4}$$

The relative entropy is defined as the difference between the vacuum-subtracted modular energy and the vacuum-subtracted entropy:

$$S_{\text{rel}} = \Delta\langle K \rangle - \Delta S. \tag{6.2.5}$$

For the regions we are talking about, the entropy of the vacuum is stationary and so drops out when we take derivatives of S_{rel}. Then for a one-parameter family of regions we have the relations

$$\frac{dS_{\text{rel}}}{d\lambda} = -\int d^{d-2}y \left[\frac{\delta S}{\delta V(y)} + 2\pi \int_{V(y)}^{\infty} dv \; \langle T_{vv} \rangle \right] \dot{V}(y), \tag{6.2.6}$$

$$\frac{d^2 S_{\text{rel}}}{d\lambda^2} = \int d^{d-2}y \; \left(2\pi \langle T_{vv} \rangle - S''_{vv} \right) \dot{V}(y)^2 - \int d^{d-2}y d^{d-2}y' \; \frac{\delta^2 S^{\text{od}}}{\delta V(y) \delta V(y')} \dot{V}(y) \dot{V}(y'). \tag{6.2.7}$$

Note here that our conjectured equation (6.1.2) can be restated as saying that the diagonal second variation of the relative entropy is zero. These equations will be mirrored holographically in Section 6.3 below.

The Bulk Setup

While we have a few remarks on the free-field and weakly-interacting cases in Appendix G, most of our nontrivial evidence for (6.1.2) and (6.1.7) comes from holography. In this section we will describe the holographic setup for the calculations outlined above. We are actually able to do without much of this machinery in Section 6.3, though it will become important afterward.

The boundary theory is a quantum field theory in d-dimensional Minkowski space obtained by deforming a CFT with relevant couplings. We take the bulk metric to be in Fefferman-Graham gauge (at least near the boundary) and choose to set the AdS length to one:

$$ds^2_{d+1} = \frac{1}{z^2}\left(dz^2 - 2du dv + d\vec{y}^2_{d-2} + \gamma_{\mu\nu} dx^\mu dx^\nu\right). \tag{6.2.8}$$

Here x^μ stands for u, v, or y. In the small-z expansion, the metric $\gamma_{\mu\nu}$ is given by [87][3]

$$\gamma_{\mu\nu} = \sum_\alpha \gamma^{(\alpha)}_{\mu\nu} z^\alpha \tag{6.2.9}$$

The term at order z^d, $\gamma^{(d)}_{\mu\nu}$, contains information about $\langle T_{\mu\nu}\rangle$ [73]. We will review the dictionary below. The terms at lower orders than z^d are associated with low-dimension operators in the theory [87]. If $\mathcal{O}$ is a relevant operator of dimension Δ and coupling g, then possible such terms that we need to be aware of include

$$\langle \mathcal{O}\rangle^m \eta_{\mu\nu} z^{m\Delta}, \qquad g^m \eta_{\mu\nu} z^{m(d-\Delta)}, \qquad g\langle \mathcal{O}\rangle \eta_{\mu\nu} z^d, \tag{6.2.10}$$

with $m \geq 2$. The coupling g, when present, is a constant. With only a single operator, terms involving derivatives of $\mathcal{O}$ will always be of higher order than z^d as long as the unitarity bound $\Delta > (d-2)/2$ is obeyed. When there is more than one low-dimension operator then we can also have terms with different combinatorial mixes of couplings and expectation values [114]. In this case, there could also be terms of the form

$$g_1^l \langle \mathcal{O}_2\rangle \eta_{\mu\nu} z^{l(d-\Delta_1)+\Delta_2}, \qquad g_1^l \partial_\mu\partial_\nu \langle \mathcal{O}_2\rangle z^{l(d-\Delta_1)+\Delta_2+2} \tag{6.2.11}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operators and g_1 is a relevant coupling associated to $\mathcal{O}_1$. There are other possibilities as well, but we will not need to enumerate them. In order demonstrate the cancellation of divergences explicitly in (6.1.2), we would need to make use of certain relationships among the various parts of the small-z expansion of the metric. Since there are general arguments for the finiteness of (6.1.2), we will be content to show that the leading state-dependent divergences cancel.[4] To that end, we will need the following fact. Suppose that in the sum (6.2.9) there is a term of the form $\gamma^{(\alpha)}_{\mu\nu} = \gamma^{(\alpha)}\eta_{\mu\nu}$. Then, assuming that α cannot be written as $\alpha_1 + \alpha_2$ for some other α_1, α_2 occuring in the sum, there will be another term $\gamma^{(\alpha+2)}_{\mu\nu}$ with a null-null component given by

$$\gamma^{(\alpha+2)}_{vv} = \frac{d-2}{(\alpha+2)(d-2-\alpha)}\partial_v^2 \gamma^{(\alpha)}. \tag{6.2.12}$$

[3]For the purposes of this discussion, we will assume all operators have generic scaling dimensions. In the generic case on a flat background a $\log z$ term in the metric expansion is unnecessary.

[4]In other words, we will only explicitly demonstrate the finiteness of (6.1.2) given some conditions on the operator dimensions which make the terms we display the only ones that are around.

This equation is obtained by solving Einstein's equations at small-z [73, 87]. Four-derivative terms are also possible, at order $\alpha + 4$, but if $d \leq 6$ then the unitarity bound ensures that $\alpha + 4 > d$. For simplicity we will ignore these terms, but with a little more effort they can also be accounted for.

Holographic Entropy and its Variations Our tool for computing the entropy is the Ryu-Takayanagi holographic entropy formula [126, 86] including quantum corrections [50, 45],

$$S = \frac{A_{\text{ext}}}{4G_N} + S_{\text{bulk}}. \tag{6.2.13}$$

A_{ext} refers to the area of the extremal area surface anchored to $\partial\mathcal{R}$ at $z = 0$. The dictionary for computing variations in the entropy as a function of $V(y)$ was laid out in [97] as follows. Let the bulk location of the extremal surface be given by

$$x^\mu = \bar{X}^\mu(y, z) = X^\mu(y) + z^2 X^\mu_{(2)}(y) + \cdots + z^d \log z X^\mu_{\log} + z^d X^\mu_{(d)} + \cdots , \tag{6.2.14}$$

where the log term is important for even dimensions and the in the case of relevant deformations with particular operator dimensions. $X^\mu(y)$ are the embedding functions of $\partial\mathcal{R}$ and $\bar{X}^\mu(y, z)$ satisfies the extremal surface equation,

$$\frac{1}{\sqrt{H}}\partial_\alpha\left(\sqrt{H}H^{\alpha\beta}\partial_\beta\bar{X}^\mu\right) + \Gamma^\mu_{\rho\sigma}H^{\alpha\beta}\partial_\alpha\bar{X}^\rho\partial_\beta\bar{X}^\sigma = 0, \tag{6.2.15}$$

where H is the induced metric on the extremal surface and Γ are bulk Christoffel symbols. Note that we have introduced the notation $\bar{X}^\mu$ for the bulk extremal surface coordinates which approach X^μ on the boundary. We will be interested in computing $\delta A_{ext}/\delta X^\mu(y)$, which by extremality is a pure boundary term evaluated at a $z = \epsilon$ cutoff surface:

$$\delta A_{\text{ext}} = \delta \int d^{d-2}y dz \, \sqrt{H} = -\int_{z=\epsilon} d^{d-2}y \, \sqrt{H}H^{zz}g_{\mu\nu}\partial_z\bar{X}^\mu\delta\bar{X}^\nu. \tag{6.2.16}$$

All of the factors appearing in the integrand need to be expanded in ϵ. The result will be a power series in ϵ containing divergent terms as well as finite terms:

$$\frac{\delta A_{\text{ext}}}{\delta X^\mu} = -\frac{K_\mu}{(d-2)\epsilon^{d-2}} + (\text{lower-order divergences in } \epsilon) - (d\, X^{(d)}_\mu + X^{(\log)}_\mu) + O(\epsilon). \tag{6.2.17}$$

Here K_μ is the extrinsic curvature of the entangling surface. We need to ensure that all divergences cancel or otherwise vanish in (6.1.2) and (6.1.7) in order that these be well-defined statements. So here we will explain the structure of the divergences in the entropy variations, as well as how to extract the finite part.

Null Variations First, we will consider the special case $X^\mu(y) = V(y)$, which is the relevant case for (6.1.2). If there are no terms of the form (6.2.11) in the metric, then the situation reduces to that of [97], in which it was shown that the divergent terms in (6.2.17) are absent as long as the entangling surface $\partial\mathcal{R}$ is locally constrained to lie in a null plane. If there are state-dependent terms of the form (6.2.11) in the metric, then there will be non-vanishing divergent contributions to $\delta A_{ext}/\delta V(y)$ proportional to, e.g., $g_1\partial_v\langle\mathcal{O}_2\rangle$. In general, an extra term at order z^α in the metric leads to a contribution at order $\alpha+2$ in $\bar{X}^\mu$ that we can obtain by solving (6.2.15) at small z. We only need to concern ourselves with terms that have $\alpha+2<d$, as those are the ones which lead to divergences. As mentioned above, for $d\leq 6$ the only terms in the metric at order α such that $\alpha+2<d$ are those of the form $\gamma^{(\alpha)}_{\mu\nu} = \gamma^{(\alpha)}\eta_{\mu\nu}$. After solving the extremal surface equation in the presence of such a term we find

$$(\alpha+2)(\alpha+2-d)X^\mu_{(\alpha+2)} = \frac{2(d-2)-\alpha d}{2(d-2)}K^\mu\gamma^{(\alpha)} + \frac{d-2}{2}\partial^\mu\gamma^{(\alpha)}. \tag{6.2.18}$$

Plugging this in to (6.2.16) leads to

$$\frac{\delta A_{\text{ext}}}{\delta V(y)} = \frac{d-2}{2(d-2-\alpha)\epsilon^{d-2-\alpha}}\partial_v\gamma^{(\alpha)}(y) + dU_{(d)}(y), \tag{6.2.19}$$

where we have eliminated a potential log term by restricting ourselves to the case of generic operator dimensions. The non-generic case can be recovered later as a limit. Using this, we can find the leading-order contribution to the second variation of the entropy:

$$\frac{\delta^2 S}{\delta V(y)\delta V(y')} = \frac{d-2}{8G_N(d-2-\alpha)\epsilon^{d-2-\alpha}}\partial_v^2\gamma^{(\alpha)}(y)\delta^{(d-2)}(y-y') + \frac{d}{4G_N}\frac{\delta U_{(d)}(y)}{\delta V(y')}. \tag{6.2.20}$$

Even though this is a very complicated object in general, we will be able to extract the δ-function contribution and see that it is given by $\langle T_{vv}\rangle$ as in (6.1.2).

Non-Null Variations When considering non-null deformations in Section 6.5 we will lose some of the special simplifications present in the null case. In that section we will only consider surfaces which are locally planar prior to being deformed, which simplifies some of the geometric expressions. More importantly, however, notice that (6.1.7) only makes reference to the vacuum-subtracted entropy variation, $\Delta S''_{\mu\nu}$, and not $S''_{\mu\nu}$ itself. So any state-independent terms in (6.2.17) can be ignored. Furthermore, we are only going to consider CFTs without any relevant deformations turned on. That means terms like (6.2.11) will not be present in the metric, and so there are no state-dependent entropy divergences. Thus for our analysis of non-null deformations we can use the formula

$$\frac{\delta^2\Delta S}{\delta X^\mu(y)\delta X^\nu(y')} = -\frac{d}{4G_N}\Delta\left(\frac{\delta X^{(d)}_\mu(y)}{\delta X^\nu(y')}\right). \tag{6.2.21}$$

Identification of the Stress Tensor We will also need a holographic formula for the stress tensor, $\langle T_{\mu\nu}\rangle$. Normally a renormalization procedure is required to define a finite stress tensor. Since our conjectures (6.1.2) and (6.1.7) are meant to be finite equations, it will be enough to regulate the stress tensor with a cutoff as we did with the entropy above.[5]

By definition, the (regulated) stress tensor is computed as the derivative of the regulated action:

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\frac{\delta I_{\text{reg}}}{\delta g^{\mu\nu}} - (\text{vacuum energy}) \ . \tag{6.2.22}$$

In holography, the regulated action is defined as the action of the bulk spacetime within the $z = \epsilon$ cutoff surface, plus additional boundary terms (like the Gibbons-Hawking-York term) which are necessary to make the variational principle well-defined. [73, 94]. For Einstein gravity in the bulk with minimally-coupled matter fields, the regulated stress tensor is then given by the Brown-York stress tensor evaluated on the $z = \epsilon$ cutoff surface [10]:[6]

$$\begin{aligned}\frac{2}{\sqrt{g}}\frac{\delta I_{\text{reg}}}{\delta g^{\mu\nu}} &= \frac{-1}{8\pi G_N \epsilon^{d-2}}\left(K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}(x,\epsilon)\right)\\ &= \frac{-1}{8\pi G_N \epsilon^{d-2}}\left(-\frac{1}{2\epsilon}\partial_\epsilon \gamma_{\mu\nu}(x,\epsilon) + \frac{1}{2\epsilon}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\epsilon\gamma_{\rho\sigma}(x,\epsilon) + \frac{1-d}{\epsilon^2}\eta_{\mu\nu}\right)\end{aligned} \tag{6.2.23}$$

Any state-dependent terms in the metric that occur at order z^α with $\alpha < d$ will contribute to divergences in the stress tensor. In particular, when we discuss null variations we will find contributions from terms of the form (6.2.12). In total we find

$$\begin{aligned}\langle T_{vv}\rangle &= \frac{\alpha+2}{16\pi G_N \epsilon^{d-2-\alpha}}\gamma_{vv}^{(\alpha+2)} + \frac{d}{16\pi G_N}\gamma_{vv}^{(d)}\\ &= \frac{d-2}{16\pi G_N (d-2-\alpha)\epsilon^{d-2-\alpha}}\partial_v^2\gamma^{(\alpha)} + \frac{d}{16\pi G_N}\gamma_{vv}^{(d)}.\end{aligned} \tag{6.2.24}$$

In the second line we used (6.2.12). Comparing this to (6.2.20), we see that the divergences indeed cancel out in (6.1.2).

For the non-null case we have additional difficulties. One can easily see that, in general, there are state-dependent divergences in $\langle T_{\mu\nu}\rangle$ that do not appear in $S''_{\mu\nu}$. Even in a CFT, if there are operators of dimension $\Delta < d/2$ then there will be a term in of $\gamma_{\mu\nu}$ at order $z^{2\Delta}$ proportional to $\langle \mathcal{O}\rangle^2\eta_{\mu\nu}$. By the unitary bound, $2\Delta > d-2$, such a term will not contribute divergences to $S''_{\mu\nu}$, but it will contribute divergences to the stress-tensor of the form

$$\langle T_{\mu\nu}\rangle\,|_{\epsilon^{2\Delta-d}} \propto \epsilon^{2\Delta-d}\langle\mathcal{O}\rangle^2\eta_{\mu\nu}. \tag{6.2.25}$$

[5] We still want to define the stress tensor so that $\langle T_{\mu\nu}\rangle = 0$ in vacuum, so the constant vacuum energy term will be subtracted.

[6] Care must be taken to impose the correct boundary conditions at $z = \epsilon$. Since we are interested in a flat-space result, we must place a flat metric boundary condition at $z = \epsilon$ before taking $\epsilon \to 0$. This is the only way to get the divergences to cancel out properly between the entropy and the energy in (6.1.2), and this treatment of the boundary condition is especially important if one wants to extend the analysis to curved space [5].

Thus, when we derive relationship (6.1.7) in Section 6.5, we will put sufficient restrictions on the theory and the states in consideration so that both sides of the equality are finite and well-defined. We have already shown above that $\Delta S''_{\mu\nu}$ is finite in a CFT, and here we find that $\langle T_{\mu\nu}\rangle$ will be finite as long as all operators of dimension $\Delta < d/2$ have vanishing expectation values, at least locally near the entangling surface. When this is true, the regularized stress tensor will be finite and equal to the standard renormalized stress tensor. Since we are also restricting ourselves to CFTs when discussing non-null variations, we can also use tracelessness of the stress tensor to simplify the answer further. The end result is the standard formula familiar from holographic renormalization [73]:

$$\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G_N}\gamma^{(d)}_{\mu\nu}. \tag{6.2.26}$$

6.3 Null Deformations and Perturbative Geometry

In this section we will prove the relation $S''_{vv} = 2\pi\langle T_{vv}\rangle$ for states with geometries corresponding to perturbations of vacuum AdS where it suffices to work to linear order in the metric perturbation. This includes classical as well as quantum states. Below in Section 6.4 we will extend our results to non-perturbative geometries.

The arguments presented here can be repeated for linearized perturbations to a non-AdS vacuum, i.e., the vacuum of a non-CFT. We restrict ourselves to the AdS case because explicit solutions to the equations are available, and the AdS case also suffices for nearly all applications in the following sections. We will see in Section 6.4 that in certain situations appeal to the non-AdS vacuum case is necessary, but because of general arguments (like the known form of the modular Hamiltonian as discussed in the Introduction) we know that it should not behave differently than the AdS case.

Bulk and Boundary Relative Entropies

In [92] it was argued that bulk and boundary relative entropies are identical:

$$S_{\text{rel}} = S_{\text{rel},bulk}, \tag{6.3.1}$$

where $S_{\text{rel},bulk}$ is calculated using the bulk quantum state restricted to the entanglement wedge of the boundary region $\mathcal{R}$ — the region of the bulk bounded by the extremal surface and $\mathcal{R}$.[7]

We already discussed in Section 6.2 the form of S_{rel} for the regions we are considering, but to leading order in bulk perturbation theory there is an analogous simple formula for $S_{\text{rel},bulk}$. We only need to know two simple facts. First, if $\partial\mathcal{R}$ is restricted to lie in the $u = 0$ plane on the boundary then, to leading order, the extremal surface in the bulk also lies in

[7] At higher orders in $1/N$ this equation is corrected [50, 43, 47]. We will not go into these corrections in detail, but will make a few comments below in Section 6.6.

the $u = 0$ plane. Second, to leading order the bulk modular energy corresponding to such a region is given by the AdS analogue of (6.2.4):

$$\Delta K_{\text{bulk}} = 2\pi \int \frac{dz d^{d-2}y}{z^{d-1}} \int_{\bar{V}(y)}^{\infty} dv \; (v - \bar{V}(y,z)) \langle T_{vv}^{\text{bulk}} \rangle. \tag{6.3.2}$$

In keeping with our earlier notation, $\bar{V}(y,z)$ gives the location of the bulk extremal surface with $\bar{V}(y, z=0) = V(y)$. Now we simply solve (6.3.1) for the vacuum-subtracted boundary entropy ΔS,

$$\Delta S = \Delta\langle K \rangle - \Delta\langle K_{\text{bulk}} \rangle + \Delta S_{\text{bulk}}, \tag{6.3.3}$$

and take two derivatives with respect to a deformation parameter λ to find

$$\frac{d^2 S}{d\lambda^2} = 2\pi \int d^{d-2}y \; \langle T_{vv} \rangle \dot{V}^2 - 2\pi \int \frac{dz d^{d-2}y}{z^{d-1}} \; \langle T_{vv}^{\text{bulk}} \rangle \dot{\bar{V}}^2 + \frac{d^2 S_{\text{bulk}}}{d\lambda^2}. \tag{6.3.4}$$

The first term represents a contribution of $2\pi\langle T_{vv}\rangle$ to S''_{vv}. So (6.1.2), $S''_{vv} = 2\pi\langle T_{vv}\rangle$, amounts to showing that the remaining two terms do not contribute to S''_{vv}. We examine them both in the next section.

Proof of the Conjecture

From the discussion around (6.3.4), the conjecture $S''_{vv} = 2\pi\langle T_{vv}\rangle$ amounts to the statement that the terms

$$-2\pi \int \frac{dz d^{d-2}y}{z^{d-1}} \; \langle T_{vv}^{\text{bulk}} \rangle \dot{\bar{V}}^2 + \frac{d^2 S_{\text{bulk}}}{d\lambda^2}. \tag{6.3.5}$$

do not contribute a δ-function to the second variation of S. Together these terms comprise the second derivative of the bulk relative entropy. We treat the two terms individually.

Bulk Modular Energy The modular energy term is simple to evaluate. Note that (6.3.2) depends on the entangling surface $V(y)$ through the extremal surface $\bar{V}(y,z)$. So functional derivatives of that expression with respect to $V(y)$ involves factors of $\delta\bar{V}(y,z)/\delta V(y')$. This is the boundary-to-bulk propagator of the extremal surface equation in pure AdS. The result, which can be extracted from our discussion in later sections, is [118]

$$\frac{\delta\bar{V}(y,z)}{\delta V(y)} = \frac{2^{d-2}\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}} \frac{z^d}{(z^2 + (y-y')^2)^{d-1}}. \tag{6.3.6}$$

Then we have

$$\frac{\delta^2 K_{\text{bulk}}}{\delta V(y_1)\delta V(y_2)} = 2\pi \left(\frac{2^{d-2}\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}}\right)^2 \int \frac{dz d^{d-2}y}{z^{d-1}} \langle T_{vv}^{\text{bulk}} \rangle \frac{z^{2d}}{(z^2 + (y-y_1)^2)^{d-1}(z^2 + (y-y_2)^2)^{d-1}} \tag{6.3.7}$$

We can diagnose the presence of a δ-function by integrating with respect to y_1 over a small neighborhood of y_2. If the result remains finite as the size of the neighborhood goes to zero, then we have a δ-function. Whether or not this happens depends on the falloff conditions on $\langle T_{vv}^{\text{bulk}} \rangle$ near $z = 0$, which in turn depends on the matter content of the bulk theory. If we suppose $\langle T_{vv}^{\text{bulk}} \rangle \sim z^{2\Delta}$ as $z \to 0$, then it is easy to see that there is no δ-function so long as

$$\Delta > (d-2)/2. \tag{6.3.8}$$

For scalar fields in the bulk, $T_{vv}^{\text{bulk}} \sim (\partial_v \phi)^2 \sim z^{2\Delta}$ where Δ is the dimension of the dual operator. This is even true when the non-normalizable mode $\phi \sim g z^{d-\Delta}$ is turned on, as long as the coupling g is constant. In the case where $\Delta = (d-2)/2$ we may find a δ-function contribution, but such a matter field would correspond to an operator saturating the unitarity bound in the CFT. These operators correspond to free fields. In a free theory there will be extra contributions to S''_{vv} besides $2\pi\langle T_{vv}\rangle$, as discussed in the Introduction and in more detail in Appendix G, so in fact this is an expected feature. For operators which do not saturate the unitarity bound, we have shown that ΔK_{bulk} does not contribute to S''_{vv}.

Bulk Entropy It is much more difficult to make statements about $d^2 S_{\text{bulk}}/d\lambda^2$. In a coherent bulk state we know that $d^2 S_{\text{bulk}}/d\lambda^2 = 0$, so for that class of states we are done.[8] More generally, we can write

$$\frac{\delta^2 S_{\text{bulk}}}{\delta V(y_1)\delta V(y_2)} = \left(\frac{2^{d-2}\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}}\right)^2 \int d^{d-2}y\,dz\,d^{d-2}y'\,dz' \frac{\delta S_{\text{bulk}}}{\delta \bar{V}(y,z)\bar{V}(y',z')} \frac{(zz')^d}{(z^2+(y-y_1)^2)^{d-1}(z'^2+(y'-y_2)^2)^{d-1}} \tag{6.3.9}$$

and ask what sort of behavior would be required of $\delta^2 S_{\text{bulk}}/\delta\bar{V}(y,z)\bar{V}(y',z')$ in order to lead to a δ-function in $y_1 - y_2$.

As a toy model, we can imagine a collection of particles on the $u = 0$ surface which are entangled in a way that depends on their distance from each other. This is a fairly general ansatz for the state of a free theory in the formalism of null quantization [139]. At small z (which is the dominant part for our calculation) this would correspond to a second variation of the form

$$\frac{\delta S_{\text{bulk}}}{\delta\bar{V}(y,z)\bar{V}(y',z')} \sim \frac{(zz')^\Delta}{(zz')^{d-1}} F\left(\frac{zz'}{(z-z')^2+(y-y')^2}\right). \tag{6.3.10}$$

The factor $(zz')^\Delta/(zz')^{d-1}$ reflects that entropy variations should be proportional to the amount of matter present at locations z and z'. The numerator encodes the falloff conditions on the density of particles in a way that is consistent with the falloff conditions on the matter

[8] In this section we treat the bulk matter fields as free. If we turn on weak interactions, then the comments of Appendix G apply. Qualitatively nothing changes.

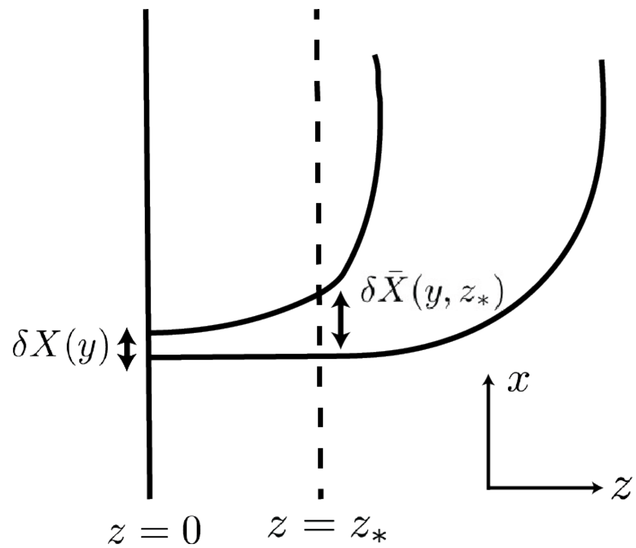

Figure 6.2: By restricting attention to $z < z_*$ the geometry is close to pure AdS, and we can solve for $\delta\bar{X}$ perturbatively. All of the $z < z_*$ data imprints itself as boundary conditions at $z = z_*$. We show that these boundary conditions are unimportant for our analysis, which means that a perturbative calculation is enough.

field, and the denominator is a measure factor that converts coordinate areas to physical areas. The function F is arbitrary.

With the assumption of (6.3.10), a constant rescaling of all coordinates by α leads to an overall factor of $\alpha^{4-2d+2\Delta}$ in (6.3.9). A δ-function in $y_1 - y_2$ would scale like α^{2-d}, and anything that scales with a power of α less than $2-d$ would correspond to a more-divergent distribution, like the derivative of a δ-function. As long as $\Delta > (d-2)/2$ this is avoided, and a δ-function is only present when the unitarity bound $\Delta = (d-2)/2$ is saturated. This is consistent with what we found previously for the modular energy, and with our general expectations for free theories.

6.4 Non-Perturbative Bulk Geometry

Now we turn to a proof that applies for a general bulk geometry, still restricting the deformations to be null on the boundary. We will use the techniques outlined in Section 6.2, which relate the entropy variations to changes in the bulk extremal surface location. At first we will stick to boundary regions where $\partial\mathcal{R}$ is restricted to a null plane, leaving a generalization to regions where $\partial\mathcal{R}$ only satisfies certain local conditions for Section 6.6.

Extremal Surface Equations

Small z, Large k The extremal surface equation (6.2.15) for $\bar{U}$ and $\bar{V}$ is a very complicated equation. If we perturb the boundary conditions by taking $V \to V + \delta V$, then the responses $\delta\bar{U}$ and $\delta\bar{V}$ will satisfy the linearized extremal surface equation, which is a bit simpler. It may be that the coordinates we have chosen are not well-suited to describing the surface perturbations deep into the bulk. That problem is solved by only aiming to analyze the equations in the range $z < z_*$ for some small but finite z_*. In fact, by choosing z_* small enough we can say that the spacetime is perturbatively close to vacuum AdS, with the perturbation given by the Fefferman-Graham expansion (6.2.9). Since the corrections to the vacuum geometry are small when z_* is small, the extremal surface equation reduces to the vacuum extremal surface equation plus perturbative corrections. All of the deep-in-the-bulk physics is encoded in boundary conditions at $z = z_*$. The situation is illustrated in Fig. 6.2

The boundary conditions at $z = z_*$ are essentially impossible to find in the general case, so the restriction to $z < z_*$ does not make the problem of finding the extremal surface any easier. However, according to (6.2.20) all we are interested in is the δ-function part of $\delta U_{(d)}$. It will turn out that this quantity is actually independent of those boundary conditions.

The idea is very simple. In Fourier space a δ-function has constant magnitude. That means it does not go to zero at large values of k, unlike the Fourier transform of a smooth function. So the strategy will be to analyze the extremal surface equation in Fourier space at large k. We will see that the large-k response of $\bar{U}$ (and hence $U_{(d)}$) is completely determined by near-boundary physics, and in particular will match the results we found in previous sections. This will establish that $S''_{vv} = 2\pi\langle T_{vv}\rangle$ for very general bulk states.

Integral Equation for $\bar{U}$ We will begin by finding an integral equation for $\bar{U}$ in the range $z < z_*$. Since $\bar{U}$ vanishes at $z = 0$ it must remain small throughout $z < z_*$, as long as z_* is small enough, and so we can use perturbation theory to find $\bar{U}$ in that range. Then we will compute the response of $\bar{U}$ to variations of the boundary conditions V at $z = 0$. Expanding (6.2.15) in small z, we can write the equation for $\bar{U}$ as

$$\partial_a^2\bar{U} + \partial_z^2\bar{U} + \frac{1-d}{z}\partial_z\bar{U} = J[\gamma_{\mu\nu}, \bar{V}, \bar{U}], \tag{6.4.1}$$

where $\gamma_{\mu\nu}/z^2$ is the deviation of the metric from vacuum AdS, as in (6.2.9). To solve this equation perturbatively we require a Green's function $G(z, y|z', y')$ of the linearized extremal surface equation that vanishes when $z = 0$ or $z = z_*$. Then the solution to (6.4.1) can be written as

$$\bar{U}(y,z) = \int \frac{d^{d-2}y'}{z_*^{d-1}}\ \partial_{z'}G(y,z|y',z_*)\bar{U}(y',z_*) + \int_{z<z_*} \frac{d^{d-2}y'dz'}{z'^{d-1}}G(y,z|y',z')J(y',z') \tag{6.4.2}$$

It is important to remember that $J(y,z)$ is itself a functional of $\bar{U}$, and the usual methods of perturbation theory would involve solving for $\bar{U}$ iteratively. It will be more useful for us to look at the Fourier transform of this equation:

$$\bar{U}(k,z) = z_*^{1-d}\partial_{z'}G_k(z|z_*)\bar{U}(k,z_*) + \int_0^{z_*} \frac{dz'}{z'^{d-1}} G_k(z|z')J(k,z'). \tag{6.4.3}$$

The Green's function with the correct boundary conditions is easily obtained from the standard Green's function $G^{\mathrm{A}dS}$ by adding a particular solution of the vacuum extremal surface equation. In Fourier space, the answer is

$$G_k(z|z') = G_k^{\mathrm{A}dS}(z|z') + (zz')^{d/2} I_{d/2}(kz) I_{d/2}(kz') \frac{K_{d/2}(kz_*)}{I_{d/2}(kz_*)} \tag{6.4.4}$$

where

$$G_k^{\mathrm{A}dS}(z|z') = -\begin{cases}(zz')^{d/2} I_{d/2}(kz)K_{d/2}(kz'), & z < z', \\ (zz')^{d/2} I_{d/2}(kz')K_{d/2}(kz), & z > z'.\end{cases} \tag{6.4.5}$$

In the limit of large k, the first term of (6.4.3) becomes exponentially suppressed. So we see that the boundary conditions at $z = z_*$ do not matter. Furthermore, the integration range $z' \gtrsim 1/k$ in the second term also becomes exponentially suppressed. So only the small-z part of the source J contributes at leading order in the large-k limit.

Terms in the Source

Let us consider the form of the source in position space in more detail. We know that $J = J[\bar{U}, \bar{V}, \gamma]$ is a functional of the extremal surface coordinates and the metric perturbation. We can treat J as a double power series in γ and $\bar{U}$ since we are doing perturbation theory in those two parameters. We will repeatedly take advantage of the "boost" symmetry of the equation: under the coordinate transformation $u \to \alpha u$, $v \to \alpha^{-1} v$, the source must transform as $J \to \alpha J$ in order for the whole equation to be covariant. Since every occurrence of $\bar{V}$ must be accompanied by either a γ or $\bar{U}$ to preserve the boost symmetry, $J[\bar{U}, \bar{V}, \gamma]$ is actually a triple power series in all three of its parameters. Another important fact is dimensional analysis, which comes from scaling all coordinates together: J has length dimension -1, while $\bar{U}$ and $\bar{V}$ have dimension 1 and γ has dimension zero. This will also be used to restrict the types of terms we can find.

The variation $\delta\bar{U}$ satisfies an integral equation similar that of $\bar{U}$ except with the source, J, replaced by the variation of the source, δJ. Like J, we can treat δJ as a power series. Each term in the δJ power series contains a single $\delta\bar{U}$, $\delta\gamma$, or $\delta\bar{V}$, multiplied by some number of $\bar{U}$, $\bar{V}$, and γ factors (and their derivatives). It is important to note that these unvaried $\bar{U}$, $\bar{V}$, and γ factors are smooth, and therefore their Fourier transforms decay at large k. So the Fourier transform of a term in δJ looks schematically like

$$\delta J(k) \sim \int_{k' << k} dk'\ h(k') \delta\Psi(k-k'), \tag{6.4.6}$$

where Ψ is either γ, $\bar{V}$, $\bar{U}$, or their derivatives and h is the Fourier transform of a smooth function. The k-dependence at large k of a given term in δJ is completely determined by the factor $\delta\Psi$ being varied. The case where $\Psi = \gamma$ can be reduced immediately to the other two, because $\delta\gamma = \delta\bar{V}\partial_v\gamma + \delta\bar{U}\partial_u\gamma$.

In Fourier space, we can write $\delta J(k,z)$ as a sum of terms of the form $\delta J_{mn} z^m k^n$ at small z and large k.[9] Since the effect of z_* is exponentially suppressed at large k, we can drop the first term in (6.4.3) push the limit in the second term off to infinity. Additionally, the difference between $G_k(z|z')$ and $G_k^{\mathrm{AdS}}(z|z')$ is exponentially suppressed. Thus for our purposes we have

$$\delta\bar{U}(k,z) = \sum_{m,n}\int_0^\infty G_k^{\mathrm{Ad}S}(z|z')\delta J_{mn} z^m k^n + O(e^{-kz_*}) \tag{6.4.7}$$
$$= \sum_{m,n}\delta J_{mn}\left(\frac{k^n z^{2+m}(d-2(m+2))}{d(m+2)(d-m-2)} - z^d 2^{m-d} k^{n-m-2+d}\frac{\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(\frac{m-d+2}{2}\right)}{\Gamma(1+d/2)}\right) + \mathcal{O}(z^{d+1})$$

If $m < d-2$ then the first term in (6.4.7) represents a contribution to the $\bar{U}$ that could have been obtained by doing the small-z expansion of the Fefferman-Graham equation. In a CFT these would consist only of geometric terms that depend on extrinsic curvatures of the entangling surface, but our boundary condition $U = 0$ guarantees that those vanish. Still, when a relevant deformation is turned on there may be terms proportional to $g_1^l \partial_v \langle\mathcal{O}_2\rangle$ which enter $\bar{U}$ at low orders in z. An important fact, enforced by the unitarity bound, is that these low-order terms are all linear in expectation values. When $m = d-2$ each of the terms in (6.4.7) becomes singular, but actually the combination above remains finite and generates at $z^d \log z$ term. Since (6.4.7) is well-behaved in this limit, we can treat the non-generic case $m = d-2$ as a limiting case of generic m. Thus throughout our discussion below m is assumed to be generic. Finally, for $d > 6$ another term proportional to z^{4+m} (and z^{6+m} in $d > 8$, etc.) should be included, but for simplicity we have not written it down. Qualitatively it has the same properties as the z^{2+m} term.

Our focus is on the z^d term, as this is where the finite contributions to the entropy variation come from, as in (6.2.20). From (6.4.7), we see that the δ-function is determined by source terms with $n-m = 2-d$, which corresponds to k^0 behavior at large k. So our task is simply to enumerate the possible terms in δJ which have this behavior. We will see that such terms are completely accounted for by the linearized analysis of the previous section[10], which completes the proof.

[9] There may also be terms in the source of the form $z^m \log(z)$. Qualitatively these terms behave similarly to the z^m terms as far as the δ-function part of the entropy variation is concerned, so we will not explicitly keep track of them.

[10] As mentioned in the previous section, for simplicity or presentation we are performing our perturbation theory around vacuum AdS, whereas in complete generality one would want to perform the analysis based around the vacuum of the theory in question. The difference is that some terms which are linear in expectation values $\langle\mathcal{O}\rangle$ might appear at higher orders in perturbation theory around AdS even though they are fully accounted for in the linearized analysis about the correct vacuum.

Ingredients Before diving into the terms of the source, we will collect all of the facts we need about the function $\bar{U}$, $\bar{V}$, γ, and their variations. In particular, we will need to know what powers of k and z we can expect them to contribute to the source.

We begin with $\bar{V}$. Unlike $\bar{U}$, $\bar{V}$ does not have any particular boundary condition at $z = 0$. Thus the Fefferman-Graham expansion for $\bar{V}$ contains low powers of z that depend on geometric data of the entangling surface. In particular, the boundary condition itself enters $\bar{V}$ at order z^0, which is neutral in terms of the $n - m$ counting. That same behavior extends to the variation $\delta\bar{V}$: in Fourier space, the state-independent parts of $\delta\bar{V}$ are functions of the combination kz. In other words, we find schematically

$$\delta\bar{V} \sim (1 + k^2 z^2 + k^4 z^4 + \cdots)\delta V. \tag{6.4.8}$$

The boundary condition δV itself is taken to go like k^0 at large k (i.e., a δ-function variation). So in terms of our power counting, which only depends on $n - m$, these terms are all completely neutral. So a factor of $\delta\bar{V}$ in the source is "free" as far as the power counting is concerned. There will be other terms in $\delta\bar{V}$, even at low powers of z, but the terms in (6.4.8) are the ones which dominate the $n - m$ counting.

$\bar{U}$ is also an extremal surface coordinate, but it has the restricted boundary condition $U = 0$. That means it does not possess terms like those in (6.4.8). The lowest-order-in-z terms that can be present are of the form $g_1^l \partial_v \langle \mathcal{O}_2 \rangle z^{2+l(d-\Delta_1)+\Delta_2}$. It is only terms like this which are linear in $\langle O \rangle$ that can show up at lower orders than z^d, because of the unitarity bound $\Delta > (d-2)/2$ Taking a variation, we find a term in $\delta\bar{U}$ of the form

$$\delta\bar{U} \sim g_1^l \partial_v^2 \langle \mathcal{O}_2 \rangle \delta V z^{2+l(d-\Delta_1))+\Delta_2}, \tag{6.4.9}$$

which has $n - m = -(2 + l(d - \Delta_1) + \Delta_2)$.

The final ingredient is the metric perturbation γ. We don't have to consider variations of γ directly, since they can be re-expressed in term s of variations of $\bar{U}$ and $\bar{V}$. γ itself has a Fefferman-Graham expansion which in includes information about the stress tensor at order z^d, but can have lower-order terms as well that depend on couplings and expectation values of operators. We will see that the important terms in the source that affect the δ-function response are those which are linear in γ.

Terms with $\delta\bar{U}$ Now we will analyze the possible terms in the source which can be obtained by piecing together the above ingredients. We begin with terms proportional to $\delta\bar{U}$. As stated above, there are dominant contributions to $\bar{U}$ in terms of the $n - m$ counting which are proportional to derivatives of expectation values of operators.

But $\bar{U}$ does not occur alone in the source J: since all terms with $\bar{U}$ alone in the equation of motion are part of the linearized equation of motion on the left-hand-side of (6.4.1). An additional factor of $\bar{V}$ does not affect the dominant $n - m$ value of the term, but the combination $\bar{U}\bar{V}$ is also prevented from appearing in J by boost symmetry. We need to have at least another factor of $\bar{U}$, or else a factor of γ. The dominant possibility without using γ is something of the form $\partial\bar{U}\partial\bar{V}\partial^2\delta\bar{U}$, where derivatives have been inserted to enforce the correct total dimensionality. Taking into account the derivatives, a term like this can have at most $n - m = 3 - 2(2 + l(d - \Delta_1) + \Delta_2) < 1 - d - 2l(d - \Delta_1) < 2 - d$, using the unitarity bound. So this sort of term will not matter for the *delta*-function response.

Making use γ allows for more possibilities. Terms of the schematic form $\gamma\delta\bar{U}$ in the source can have $n - m > 2 - d$, and if we allow fine-tuning of operator dimensions we can even reach $n - m = 2 - d$. These sources are obtained by taking a state-independent term in γ which is proportional some power of g_1 and a term in $\delta\bar{U}$ which is proportional to $\partial_v^2\langle\mathcal{O}_2\rangle$. We can even multiply by more factors of γ, giving $\gamma^l\delta\bar{U}$ schematically, as well as factors of $\bar{V}$, as long as we don't involve more factors of $\bar{U}$. A second factor of $\bar{U}$ brings with it a large z-scaling, so we run into the same problem we had above in the $\bar{U}\bar{V}\delta\bar{U}$ case. The end result is that all of the potentially-important terms in this analysis are linear in the expectation value $\langle\mathcal{O}\rangle$. That means they are subject to restrictions on the modular Hamiltonian as mentioned in the Introduction, which means that they will actually not show up in (6.1.2) despite being allowed by dimensional analysis.

Terms with $\delta\bar{V}$ Now we consider terms in δJ that are proportional to a variation $\delta\bar{V}$. As discussed above, $\delta\bar{V}$ has several state-independent terms which are neutral in the $n - m$ counting. Due to the boost symmetry, $\delta\bar{V}$ cannot occur alone in δJ. It must be accompanied by at least two factors of $\bar{U}$ or one factor of γ. We have already discussed how two factors of $\bar{U}$ have a large-enough z-scaling to make the term uninteresting, so it remains to consider factors of γ.

Terms in the source proportional to $\delta\bar{V}$ with only a single factor of γ are those present in the theory of linearized gravity about vacuum AdS. Furthermore, since we argued that boundary conditions at $z = z_*$ do not affect the answer, the Green's function we use to compute the effects of the source is *also* the same as we would use in linearized gravity about vacuum AdS. We already considered the linearized gravity setup in Section 6.3, even though we didn't solve it using the methods of this section. In Section 6.3 we saw that $S''_{vv} = 2\pi\langle T_{vv}\rangle$, and so it is enough for us now to prove that the general computation of the δ-function terms reduces to the linearized gravity case. There is only one more loose end to consider: terms in δJ proportional to $\delta\bar{V}$ that have more than one factor of γ.

With more than a single factor of γ, it is clear that the only contributions that could possibly be important at large k are those coming from the powers of z less than z^d in (6.2.9). These terms are made up of couplings g, operator expectation values $\langle\mathcal{O}\rangle$, and their derivatives. In order to have the correct boost scaling, we need to include v-derivatives acting on operator expectation values. As we have discussed many times, the unitarity bound prevents any term with more than one factor of $\langle\mathcal{O}\rangle$ from being important. So just as with the $\delta\bar{U}$ terms discussed previously, all of these terms are subject to constraints from the modular Hamiltonian and hence do not appear in (6.1.2)

Our analysis so far has been very simple , but we have reached an important conclusion that bears repeating: the source terms which give the k^0 behavior for $\delta U_{(d)}$ were already present in the linearized gravity calculation of the previous section, and we are allowed to use the ordinary Green's function G^{AdS} to compute their effects. In other words, for the purpose of calculating the δ-function response we have reduced the problem to linearized gravity. We have shown previously that the linearized gravity setup leads to $S''_{vv} = 2\pi\langle T_{vv}\rangle$, and so our proof is complete.

6.5 Non-Null Deformations

Having established $S''_{vv} = 2\pi\langle T_{vv}\rangle$ for deformations of entangling surfaces restricted to lie in the plane $u = 0$, we will now analyze arbitrary deformations of the entangling surface to prove (6.1.7). The technique is very similar to that of the previous section. As discussed in Sec 6.2, there are issues related to cancellations of divergences that make this result much less universal. Thus, we will restrict attention to CFTs in states where all operators with dimension $\Delta \leq d/2$ have vanishing expectation values in some finite neighborhood of the entangling surface. These restrictions are sufficient to make (6.1.7) finite.

New Boundary Conditions

Above we analyzed deformations within the null plane $u = 0$ at small z and large k. These limits allowed us to show that the perturbation theory for $\delta U_{(d)}$ reduced to linearized gravity, which we had already studied in Section 6.3. There strategy here is the same, except we want to be able to perform perturbation theory on both $\bar{U}$ and $\bar{V}$ in order to get more than just the null-null variations. The simplest case, which is all that we will analyze in this work, is to start with the boundary condition $V = 0$ at $z = 0$ in addition to $U = 0$. In other words, we take our undeformed entangling surface to be the $v = u = 0$ plane. That is a severe restriction on the type of surface we are considering, but we gain the flexibility of being able to do perturbation theory in both $\bar{U}$ and $\bar{V}$. From (6.2.21),

$$\frac{\delta^2 \Delta S}{\delta X^\mu(y)\delta X^\nu(y')} = -\frac{d}{4G_N}\Delta\left(\frac{\delta X_\mu^{(d)}(y)}{\delta X^\nu(y')}\right), \tag{6.5.1}$$

where ΔS refers to the vacuum-subtracted entropy. Vacuum subtraction removes all state-independent terms from the entropy, including divergences.

With the $U = V = 0$ boundary conditions, we can again write down our perturbative extremal surface equation for the $z < z_*$ part of the bulk. Since the null direction is no longer preferred, we will use a covariant form of the linearized equation:

$$\partial_a^2 \bar{X}^\mu + \partial_z^2 \bar{X}^\mu + \frac{1-d}{z}\partial_z \bar{X}^\mu = J^\mu[\gamma, \bar{X}] \tag{6.5.2}$$

Following the same steps as in the previous section, we can use Green's functions to solve this equation in Fourier space. There is one new ingredient that we did not have before. When we computed the variation of $U_{(d)}$ with respect to V, we were changing the boundary conditions of $\bar{V}$ and computing the response in $\bar{U}$. In particular, the boundary condition of $\bar{U}$ itself remained zero. In the more general setup of this section, we need to compute the response of a particular component of $\bar{X}^\mu$ when its own boundary conditions at $z=0$ are varied.

Since we only care about the δ-function contribution to the entropy variation, we will immediately use $\delta X^\mu(k) = e^{iky_0}\xi^\mu$ as the boundary condition for $\delta\bar{X}^\mu$. Here ξ^μ is just a constant vector which tells us the direction of the perturbation. The presence of this boundary condition at $z=0$ is simple to account for with one additional term in the integral equation for $\bar{X}^\mu$ compared to (6.4.3) in the previous section. In total, we now have

$$\delta X^\mu(k,z) = z^{d/2}K_{d/2}(kz)\frac{dk^{d/2}}{2^{d/2}\Gamma(1+d/2)}\xi^\mu e^{iky_0} \\ + z_*^{1-d}\partial_{z'}G(z|z_*)\delta\bar{X}^\mu(k,z_*) + \int_0^{z_*}\frac{dz'}{z'^{d-1}}G_k(z|z')\delta J^\mu(k,z') \tag{6.5.3}$$

As above, in the large-k limit the term coming from boundary conditions at $z=z_*$ (the first term in the second line of (6.5.3)) will drop out and so can be ignored completely. The term from boundary conditions at $z=0$ (the first line of (6.5.3)) will not drop out automatically, and so will contribute to the second entropy variation. This contribution to the entropy variation is known as the entanglement density in the literature and was previously computed in [49, 12]. From (6.5.3) it is clear that the entanglement density is completely determined by the AdS Green's function and is therefore state-independent. By restricting attention to the vacuum-subtracted entropy the entanglement density will drop out, and in any case is not proportional to a δ-function.

Terms in the Source

As in the null deformation discussion of Section 6.4, we need to compute the effects of the source δJ^μ. As we did there, we will accomplish this by cataloging the various terms which can appear in the power series expansion of J^μ as a function of $\bar{X}$ and γ. Again, terms which scale like $k^n z^m$ ultimately lead to $k^{n-m+d-2}$ dependence at large k for $\delta X^\mu_{(d)}$. Any term in δJ^μ will look like $\delta\bar{X}^\nu$ multiplied by some function of γ and $\bar{X}$. For the purposes of computing δJ^μ only the state-independent parts of $\delta\bar{X}^\nu$, represented by the first line of (6.5.3), will matter. That is because these terms are a function of the combination kz, which means they have $n-m=0$. Now we just have to consider all of the possible combinations of γ and $\bar{X}$ which multiply $\delta\bar{X}$.

There cannot be any terms in δJ^μ that are schematically of the form $\bar{X}\delta\bar{X}$ with some derivatives but no factors of γ. Such a term would have to come from nonlinearities in the vacuum AdS extremal surface equation. That equation is invariant under $\bar{X}\to-\bar{X}$, so all terms have to have odd parity like the linear terms. Anything of the form $\bar{X}\bar{X}\delta\bar{X}$, or higher powers of $\bar{X}$, will not contribute at large k because of power counting: The vanishing boundary condition means that $\bar{X}$ starts at order z^d, which means that the most favorable possible term of this type, $(\partial_z \bar{X})^2 \partial_z^2 \delta \bar{X}$, still only amounts to a contribution to the entropy variation which scales like k^{2-d}.

Now we consider terms which have at least one factor of γ. Because we have assumed that expectation values of operators with dimension $\Delta \leq d/2$ vanish, the leading order piece of γ scales like z^d. Thus, we can easily get contributions to $\delta X_{(d)}$ which go like k^0 from source terms which are schematically of the form $\gamma \partial^2 \delta \bar{X}$, as well as other combinations. Given their importance, we will analyze terms of the form $\gamma \delta \bar{X}$ below in more detail.

Terms with additional factors of $\bar{X}$ or γ beyond the first power of γ will not lead to non-decaying behavior at large k because of power counting. So we see that only the linear gravitational backreaction is necessary to completely characterize $\Delta S''_{\mu\nu}$. We will now calculate those terms explicitly.

Linearized Geometry

We have reduced our task to computing J^μ to linear order in γ and $\bar{X}^\mu$ (the latter condition comes from our choice of a planar undeformed entangling surface). This is a simple exercise in expanding (6.2.15). The result in position space is

$$\begin{aligned} J^\mu = &-\frac{1}{2}\partial_z \gamma_{cc} \partial_z \bar{X}^\mu + \partial_a(\gamma_{ab}\partial_b \bar{X}^\mu) - \eta^{\mu\nu}\partial_z \gamma_{\nu\rho}\partial_z \bar{X}^\rho \\ &- \eta^{\mu\nu}(\partial_a \gamma_{\nu\rho} + \partial_\rho \gamma_{\nu a} - \partial_\nu \gamma_{a\rho})\partial_a \bar{X}^\rho - \frac{1}{2}\eta^{\mu\nu}(2\partial_a \gamma_{\nu a} - \partial_\nu \gamma_{aa}) - \frac{1}{2}\partial_a \gamma_{cc}\partial_a \bar{X}^\mu. \end{aligned} \tag{6.5.4}$$

a, b, c indices represent the y-directions and repeated indices are summed over. Taking the variation and evaluating at $\bar{X}^\mu = 0$ gives

$$\begin{aligned} \delta J^\mu = &-\frac{1}{2}\partial_z \gamma_{cc}\partial_z \delta \bar{X}^\mu + \partial_a(\gamma_{ab}\partial_b \delta \bar{X}^\mu) - \eta^{\mu\nu}\partial_z \gamma_{\nu\rho}\partial_z \delta \bar{X}^\rho \\ &- \eta^{\mu\nu}(\partial_a \gamma_{\nu\rho} + \partial_\rho \gamma_{\nu a} - \partial_\nu \gamma_{a\rho})\partial_a \delta \bar{X}^\rho \\ &- \frac{1}{2}\eta^{\mu\nu}(2\partial_\rho \partial_a \gamma_{\nu a} - \partial_\rho \partial_\nu \gamma_{aa})\delta \bar{X}^\rho - \frac{1}{2}\partial_a \gamma_{cc}\partial_a \delta \bar{X}^\mu. \end{aligned} \tag{6.5.5}$$

The only terms in (6.5.5) that will contribute at k^0 are those with two y derivatives acting on $\delta \bar{X}^\mu$ or with z derivatives, i.e., the first line of (6.5.5). Then the result for $\delta X^\mu_{(d)}$ at large k is obtained from (6.5.3) as

$$\begin{aligned} \delta X^\mu_{(d)}(k) &= \frac{-1}{2^{d-2}\Gamma(d/2)^2}\left[\left(\langle \gamma^{(d)\mu}_\nu \rangle + \frac{1}{2}h^{ab}\langle \gamma^{(d)}_{ab}\rangle \eta^\mu_\nu\right)\left(\lim_{z\to 0}\frac{1}{2}z^d K_{d/2}(z)^2\right)\right. \\ &\qquad \left. - \left(\eta^\mu_\nu \frac{k^a k^b}{k^2}\langle \gamma^{(d)}_{ab}\rangle\right)\left(\int_0^\infty dz z^{d+1}K_{d/2}(z)^2\right)\right] e^{iky_0}\xi^\nu \\ &= -\frac{8\pi G_N}{d}\left[\langle T^\mu_\nu \rangle + \frac{1}{2}h^{ab}\langle T_{ab}\rangle \eta^\mu_\nu - \frac{d}{d+1}\eta^\mu_\nu \frac{k^a k^b}{k^2}\langle T_{ab}\rangle\right] e^{iky_0}\xi^\nu \end{aligned} \tag{6.5.6}$$

Here we have explicitly included factors of the entangling surface metric h^{ab} (which is equal to δ^{ab}) rather than using repeated a, b indices for added clarity. In the last line, we have used the dictionary (6.2.26) to replace $\gamma^{(d)}_{\mu\nu}$ with $\langle T_{\mu\nu}\rangle$.

The first two terms of (6.5.6) correspond to δ-functions in position space. The final term clearly contains a δ-function piece which will end up being proportional to the trace of $\langle T_{ab}\rangle$, but it also contains off-diagonal contributions. We can use the identity

$$\int d^{d-2}k\ \frac{k^a k^b}{k^2} e^{ik(y-y_0)} \propto \partial_a \partial_b \frac{1}{|y-y_0|^{d-4}} \propto \frac{\delta_{ab} - (d-2)(y-y_0)^a (y-y_0)^b/(y-y_0)^2}{|y-y_0|^{d-2}}. \tag{6.5.7}$$

to see the full effect in position space. However, for our purposes we are only interested in the δ-function contribution. Isolating this part and combining it with the first two terms of (6.5.6), we ultimately find

$$\Delta S''_{\mu\nu} = 2\pi \left(n^\rho_\mu n^\sigma_\nu \langle T_{\rho\sigma}\rangle + \frac{d^2-3d-2}{2(d+1)(d-2)} n_{\mu\nu} h^{ab} \langle T_{ab}\rangle \right) \tag{6.5.8}$$

where $n_{\mu\nu}$ is the normal projector of the entangling surface. This completes our derivation of (6.1.7).

6.6 Discussion

We have found formulas for the δ-function piece of the second variation of entanglement entropy in terms of the expectation values of the stress tensor. In this section we conclude by discussing a number of possible extensions and future applications of this result.

Higher Orders in $1/N$

Since we believe (6.1.7) and (6.1.2) to be valid at finite-N, it must be that our calculations are not affected by higher-order corrections holographically. There are two classes of higher-order corrections we can consider: those coming form higher-curvature corrections in the bulk, and those coming from the bulk entropy. These corrections can be encapsulated in the all-orders formula [47, 45]

$$S = S_{\text{gen}}[e(\mathcal{R})] = S_{\text{Dong}}[e(\mathcal{R})] + S_{\text{bulk}}[e(R)]. \tag{6.6.1}$$

The first term here is the Dong entropy functional [43], which is an integral of geometric data over the surface $e(\mathcal{R})$, and the second term is the bulk entropy lying within the region bounded by $e(\mathcal{R})$. Finally, the surface $e(\mathcal{R})$ is the one that extremizes the S_{gen} functional.

If we ignore the S_{bulk} term for a moment, then S_{Dong} behaves qualitatively the same way as the area in the Ryu-Takayanagi formula. The coordinates $\bar{X}^\mu$ of $e(\mathcal{R})$ obey a certain differential equation, and the variations in the entropy are still related to $\delta X^\mu_{(d)}$ as before.

One change is that the overall coefficient of $\delta X^{\mu}_{(d)}$ relative to the entropy will change in a way that depends on the bulk higher curvature couplings. However, the dictionary relating $\gamma_{\mu\nu}$ to $T_{\mu\nu}$ also changes in a way that precisely preserves (6.1.7) [5].

Incorporating the S_{bulk} term is simple in principle but difficult in practice to deal with. Since it is S_{gen} that must be extremized, we have to include an extra term in the equation of motion proportional to $\delta S_{\text{bulk}}/\delta \bar{X}^{\mu}(y)$. That means the bulk entropy itself plays a role in determining the position of the surface. It was argued in [3] (assuming some mild falloff conditions on variations of the entropy) that the presence of this source would not affect the dictionary relating $\delta X^{\mu}_{(d)}$ to the variation of the entropy. Beyond this, the most we can say about the contributions of the entropy are arguments of the type given above in Section 6.3. While this is a potential loophole in our arguments, we still believe that our evidence overwhelmingly suggests that new contributions to (6.1.7) do not appear.

Local Conditions On $\partial\mathcal{R}$ Are Enough

We now briefly discuss why we expect that we can relax the stationarity conditions on the entangling surface to hold just in the vicinity of the deformation point. We will focus on the null-null case, but a similar result should hold in the non-null case (where it should also be true that our restriction on expectation values for operators with $\Delta < d/2$ is allowed to be local).

We can analyze the source (6.4.6) in a little more detail in the case where we only impose local stationarity near $y = y_0$. Even though in position space $\bar{U}(y_0, z)$ does not contain any state-independent terms at low orders in the z-expansion near, the inherent non-locality of the Fourier transform $\bar{U}(k, z)$ will contain those terms. There are two ways this could affect (6.4.6): through $\delta\Psi = \delta\bar{U}$ or through the h-factor. In either case, the large k limit reduces to the problem back to the globally-stationary setup.

For example, by setting $\delta V(k) = e^{iky_0}$ we can isolate the part of $\delta U_{(d)}$ that gives a δ-function localized at $y = y_0$. Then the important part of $\delta\bar{V}$ (i.e., the state-independent part) is

$$\delta\bar{V}(k,z) = e^{iky_0} 2^{\frac{d-2}{2}} \Gamma(d/2)(kz)^{d/2} K_{d/2}(kz). \tag{6.6.2}$$

Then we can organize (6.4.6) as a derivative expansion of h, with the leading term given by

$$\delta J(k,z) \sim e^{iky_0} h(z, y_0)(kz)^{d/2} K_{d/2}(kz), \tag{6.6.3}$$

and the remaining terms suppressed by powers of k. In other words, the integral over k' in (6.4.6) combined with the $(k-k')$-dependence of δV essentually returns h to position space localized near $y = y_0$. Only the first d derivatives of h at $y = y_0$ will be relevant at large k, so only the first d derivatives of U need to be set equal to zero at $y = y_0$ in order for the large-k behavior to match the case where U vanishes identically. Thus it is enough to have entangling surfaces which are in the $u = 0$ plane up to order d in $y - y_0$.

Note, this crude analysis does not strictly apply if the entangling surface cannot be globally written in terms of functions $U(y), V(y)$. For example, an entangling surface which is topologically a sphere does not fall within the regime of our arguments. We leave an analysis of those types of regions for future work.

Curved Backgrounds

It is interesting to ask what happens to this proof when the boundary spacetime is curved. Our arguments make it clear that $S''_{\mu\nu}$ is completely determined by local properties of the state in the bulk and on the boundary. So naturally one would expect that there is a curved-space analogue of the same formula. In [5, 62], several local conditions on the entangling surface and spacetime curvature were found such that the QNEC would hold in curved space and be manifestly scheme-independent. We would expect that under those same conditions one could show that $S''_{vv} = 2\pi\langle T_{vv}\rangle$. Non-null variations in a curved background have yet to be explored, and it would be interesting to investigate aspects of the curved background setup in more detail.

Connections to the QFC and Gravity

An interesting application of our result is to the interpretation of Einstein's equations. Combining (6.1.7) with Einstein's equations leads to an explicit formula relating geometry to entanglement. This result is the latest in a growing trend of connections between geometry and entanglement [121, 112, 89].

We can make a more direct connection with the deep result by Jacobson of the Einstein equation of state [90]. There it was argued that Einstein's equations were equivalent to a statement of thermal equilibrium across an arbitrary local Rindler horizon, namely the equation $\delta Q = T\delta S$, together with an assumption that S is proportional to area. This argument used a thermodynamic definition of the entropy without mentioning quantum entanglement. We can give this result a modern interpretation with the equation $S''_{vv} = 2\pi\langle T_{vv}\rangle$.

The connection to our result is most easily phrased in terms of the generalized entropy for a field theory coupled to gravity, which is defined as

$$S_{\text{gen}} = \frac{A}{4G_N} + S_{\text{ren}}. \tag{6.6.4}$$

Here G_N is the renormalized Newton's constant, and S_{ren} is the renormalized entropy of the field theory system. Variations of this quantity were considered in [21], where the conjecture $S''_{\text{gen}} \leq 0$ was dubbed the Quantum Focusing Conjecture (QFC). When the entangling surface is locally flat, it was argued in [21] that[11]

$$S''_{\text{gen},vv} = -\frac{R_{vv}}{4G_N} + S''_{\text{ren},vv} \tag{6.6.5}$$

Instead of assuming the Einstein equations hold as in [21], we will instead use the result $S''_{\text{ren},vv} = 2\pi\langle T_{vv}\rangle$.[12] Then we have

$$S''_{\text{gen},vv} = -\frac{R_{vv}}{4G_N} + 2\pi\langle T_{vv}\rangle. \tag{6.6.6}$$

Now we can say that the null-null component of Einstein's equations, $R_{vv} = 8\pi G_N\langle T_{vv}\rangle$, is equivalent to the statement $S''_{\text{gen},vv} = 0$. Following [90], this is equivalent to the full Einstein equations with an undetermined cosmological constant.

The end result is that we can replace Jacobson's original assumption of $\delta Q = T\delta S$ with the statement that $S''_{\text{gen}} = 0$, together with (6.1.2).

Proof for General CFTs

We view our results as sufficient motivation to look for a proof of (6.1.7) and (6.1.2) in general field theories. In conformal field theories, entanglement entropy can be calculated using the replica trick. A replicated CFT is equivalent to a CFT with a twist defect. Within the technology of defect CFTs, shape deformations of entropy is generated by displacement operators (see [9] for a review of these concepts). The variation $\delta^2 S/\delta V(y)\delta V(y')$ then is related to the OPE structure of displacement operators in this setup. Since the coefficient of the delta function piece in (6.1.1) is fixed to have dimension d and spin 2, one might be able to see that only the stress tensor could appear as a local operator in S''_{vv}. It further needs to be shown that no other non-linear (in the state) contributions could appear in S''_{vv}. Results in that direction will be reported in future work [8].

[11] We are being somewhat cavalier about extracting the δ-function term, especially since doing so in the context of higher-curvature gravity can lead to apparent violations of the QFC [61]. It was shown in [105] that these apparent violations can be avoided so long as we smear over a Planck-sized region of the entangling surface. If the mass scales of the matter sector are all much less than the Planck scale, then we expect that a Planck-sized surface deformation should be indistinguishable from a δ-function deformation as far as the field theory is concerned.

[12] Note that we are assuming that (6.1.2) remains unaffected by background null curvature. We discussed why this is expected to be true in the previous section.

Chapter 7

Entropy Variations and Light Ray Operators from Replica Defects

7.1 Introduction

Despite much progress in understanding entanglement entropy using bulk geometric methods in holographic field theories [126, 125, 86], significantly less progress has been made on the more difficult problem of computing entanglement entropy directly in field theory. Part of what makes entanglement entropy such a difficult object to study in field theory is its inherently non-local and state-dependent nature.

One way to access the structure of entanglement in field theories is to study its dependence on the shape of the entangling surface. Such considerations have led to important results on the nature of entanglement in quantum field theories [52, 26, 96, 5, 27, 9, 6, 49, 115]. To study the shape dependence of entanglement entropy for QFTs in $d > 2$ dimensions, consider a Cauchy slice Σ containing a subregion $\mathcal{R}$ with entangling surface $\partial\mathcal{R}$ in a general conformal field theory. By unitary equivalence of Cauchy slices which intersect the same surface $\partial\mathcal{R}$, the entanglement entropy for some fixed global state can be viewed as a functional of the entangling surface embedding coordinates $X^\mu(y^i)$ where the y^i with $i = 1, ..., d-2$ are internal coordinates on $\partial\mathcal{R}$. We write:

$$S_{\mathcal{R}} = S[X(y)]. \tag{7.1.1}$$

The shape dependence of the entanglement entropy can then be accessed by taking functional derivatives. In particular, we can expand the entanglement entropy about some background entangling surface $X(y) = X_0(y) + \delta X(y)$ as

$$\begin{aligned} S[X] = S[X_0] &+ \int d^{d-2}y \, \frac{\delta S_{\mathcal{R}}}{\delta X^\mu(y)}\bigg|_{X_0} \delta X^\mu(y) \\ &+ \int d^{d-2}y d^{d-2}y' \, \frac{\delta^2 S_{\mathcal{R}}}{\delta X^\mu(y)\delta X^\nu(y')}\bigg|_{X_0} \delta X^\mu(y)\delta X^\nu(y') + \dots \, . \end{aligned} \tag{7.1.2}$$

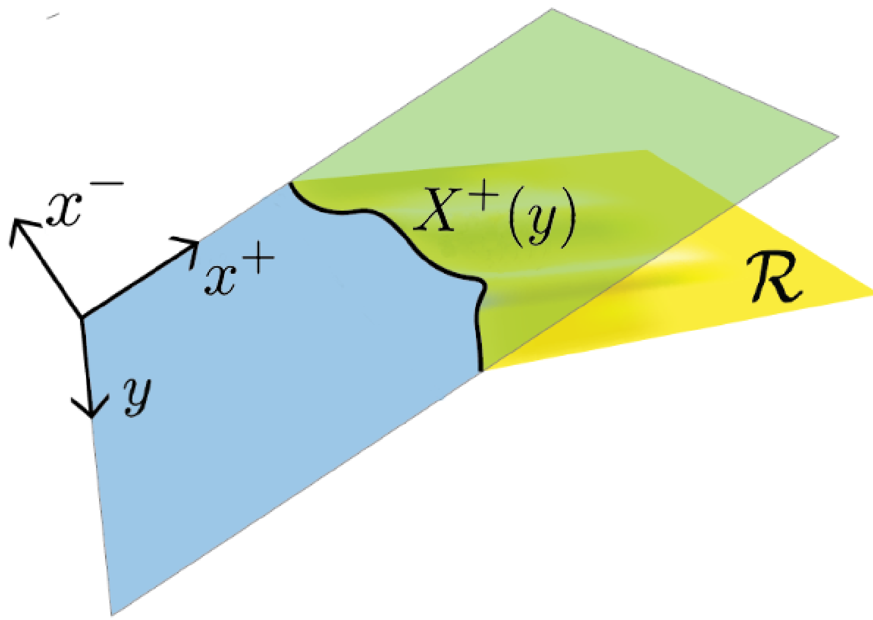

Figure 7.1: We consider the entanglement entropy associated to a spatial subregion $\mathcal{R}$. The entangling surface lies along $x^- = 0$ and $x^+ = X^+(y)$. In this work, we study the dependence of the entanglement entropy on the profile $X^+(y)$.

This second variation has received a lot of attention in part because it is an essential ingredient in defining the *quantum null energy condition* (QNEC) [21, 27]. The QNEC bounds the null-null component of the stress tensor at a point by a specific contribution from the second shape variation of the entanglement entropy. More specifically, this second variation can be naturally split into two pieces - the *diagonal* term which is proportional to a delta function in the internal coordinates y^i and the *off-diagonal* terms[1]

$$\frac{\delta^2 S_{\mathcal{R}}}{\delta X^+(y)\delta X^+(y')} = S''(y)\delta^{(d-2)}(y-y') + (\text{off-diagonal}). \tag{7.1.3}$$

where (X^+, X^-) are the null directions orthogonal to the defect. The QNEC states that the null energy flowing past a point must be lower bounded by the diagonal second variation

$$\langle T_{++}(y)\rangle \geq \frac{\hbar}{2\pi} S''(y), \tag{7.1.4}$$

[1]Note that the entanglement entropy, being UV divergent, will typically have divergent contributions that are local to the entangling surface. These will show up as a limited set of diagonal/contact terms in (7.1.3). For deformations about a sufficiently flat entangling surface these terms do not contribute to the contact term that is the subject of the QNEC. The divergent terms will not be the subject of investigation here.

where we are taking $\mathcal{R}$ to be a Rindler wedge. This inequality was first proposed as the $G_N \to 0$ limit of the quantum focussing conjecture [21], and was first proven in free and super-renormalizable field theories in [26]. The proof for general QFTs with an interacting UV fixed point was given in [9]. More recently, yet another proof was given using techniques from algebraic quantum field theory [40].

The method of proof in the free case involved explicitly computing S''_{++} where it was found that

$$S'' = \frac{2\pi}{\hbar} \langle T_{++} \rangle - Q \tag{7.1.5}$$

where for general states $Q \geq 0$. In contrast, the proof in general QFTs relied on relating the inequality (7.1.4) to the causality of a certain correlation function involving modular flow. This left open the question of whether S'' could be explicility computed in more general field theories.

In [106] the diagonal term S'' was computed in large N QFTs in states with a geometric dual. Remarkably, the result was

$$S''(y) = 2\pi \langle T_{++}(y) \rangle \tag{7.1.6}$$

where we have now set $\hbar = 1$. In other words, $Q = 0$ for such theories. In that work, it was argued that neither finite coupling nor finite N corrections should affect this formula. This led the authors of [106] to conjecture (7.1.6) for all interacting CFTs. The main goal of this paper is to provide evidence for (7.1.6) in general CFTs with a twist gap.

The method of argument will follow from the replica trick for computing entanglement entropy. The replica trick uses the formula

$$S[\mathcal{R}] = \lim_{n \to 1} (1 - n\partial_n) \log \mathrm{Tr}[\rho_{\mathcal{R}}^n] \tag{7.1.7}$$

to relate the entanglement entropy to the partition function of the CFT on a replicated manifold [83, 31] (see also [102, 129, 124, 109])

$$\mathrm{Tr}[\rho_{\mathcal{R}}^n] = Z_n/(Z_1)^n. \tag{7.1.8}$$

At integer n, Z_n can be computed via a path integral on a branched manifold with n-sheets. Alternatively, one can compute this as a path integral on an unbranched manifold but in the presence of a twist defect operator Σ_n of co-dimension 2 that lives at the entangling surface [13]. Doing so allows us to employ techniques from defect CFTs. See [15, 67, 64, 14] for a general introduction to these tools.

In particular, shape deformations of the defect are controlled by a defect operator, namely the displacement operator, with components $\hat{D}_+, \hat{D}_-$. This operator is universal to defect CFTs. Its importance in entanglement entropy computations was elucidated in [15, 9, 13]. Consequently, the second variation of the entanglement entropy is related to the two-point function of displacement operators

$$\frac{\delta^2 S}{\delta X^+(y) \delta X^+(y')} = \lim_{n \to 1} \frac{-2\pi}{n-1} \langle \Sigma_n^\psi \hat{D}_+(y) \hat{D}_+(y') \rangle \,, \tag{7.1.9}$$

where the notation Σ_n^ψ will be explained in the next section.

Since we are interested in the delta function contribution to this second variation, we can take the limit where the two displacement operators approach each other, $y \to y'$. This suggests that we should study the OPE of two displacement operators and look for terms which produce a delta function, at least as $n \to 1$.

It might seem strange to look for a delta function in an OPE since the latter, without further input, results in an expansion in powers of $|y - y'|$. We will find a delta function can emerge from a delicate interplay between the OPE and the replica limit $n \to 1$.

An obvious check of our understanding of (7.1.6) is to explain how this formula can be true for interacting theories while there exist states for which $Q > 0$ in free theories. This is a particularly pertinent concern in, for example, $\mathcal{N} = 4$ super-Yang Mills where one can tune the coupling to zero while remaining at a CFT fixed point. We will find that in the free limit certain terms in the off-diagonal contributions of (7.1.3) become more singular and "condense" into a delta function in the zero coupling limit. In a weakly interacting theory it becomes a question of resolution as to whether one considers Q to be zero or not.

In fact this phenomenon is not unprecedented. The authors of [82] studied energy correlation functions in a so called conformal collider setup. The statistical properties of the angular distribution of energy in excited states collected at long distances is very different for free and interacting CFTs. We conjecture that these situations are controlled by the same physics. Explicitly, in certain special "near vacuum" states, there is a contribution to the second variation of entanglement that can be written in terms of these energy correlation functions.

Schematically, we will find

$$\frac{\delta^2 S}{\delta X^+(y)\delta X^+(y')} - \frac{2\pi}{\hbar} \langle T_{++}\rangle \, \delta^{(d-2)}(y - y') \sim \int ds e^s \, \langle \mathcal{O}\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y')e^{iKs}\mathcal{O}\rangle \tag{7.1.10}$$

where

$$\hat{\mathcal{E}}_+(y) = \int_{-\infty}^{\infty} d\lambda \, \langle T_{++}(x^+ = \lambda, x^- = 0, y)\rangle \tag{7.1.11}$$

is the averaged null energy operator discussed in [82] and the $\mathcal{O}$'s should be thought of as state-creation operators. The operator K is the boost generator about the undeformed entangling surface.

The singularities in $|y - y'|$ of the correlator in (7.1.10) are then understood by taking the OPE of two averaged null energy operators. This OPE was first discussed in [82] where a new non-local "light ray" operator of spin 3 was found to control the small $y - y'$ limit.

In the free limit, we will show that this non-local operator has the correct scaling dimension to give rise to a new delta function term in (7.1.10). In the interacting case this operator picks up an anomalous dimension and thus lifts the delta function.

In other words, the presence of an extra delta function in the second variation of the entanglement entropy in free theories can be viewed as a manifestation of the singular behavior of the conformal collider energy correlation functions in free theories. This is just another manifestation of the important relationship between entanglement and energy density in QFT.

The presence of this spin-3 light ray operator in the shape variation of entanglement in specific states however points to an issue with our defect OPE argument. In particular one can show that this contribution cannot come directly from one of the local defect operators that we enumerated in order to argue for saturation. Thus one might worry that there are other additional non-trivial contributions to the OPE that we miss by simply analyzing this local defect spectrum. The main issue seems to be that the $n \to 1$ limit does not commute with the OPE limit. Thus in order to take the limit in the proper order we should first re-sum a subset of the defect operators in the OPE before taking the limit $n \to 1$. For specific states we can effectively achieve this resummation (by giving a general expression valid for finite $|y - y'|$) however for general states we have not managed to do this. Thus, we are not sure how this spin-3 light ray operator will show up for more general states beyond those covered by (7.1.10). Nevertheless we will refer to these non-standard contributions as arising from "nonlocal defect operators."

The basic reason it is hard to make a general statement is that entanglement can be thought of as a state dependent observable. This state dependence shows up in the replica trick as a non-trivial n dependence in the limit $n \to 1$ so the order of limits issue discussed above is linked to this state dependence. We are thus left to compute the OPE of two displacement operators for some specific states and configurations. This allows us to check the power laws that appear in the $|y_1 - y_2|$ expansion for possible saturation violations. Given this we present two main pieces of evidence that the nonlocal defect operators do not lead to violations of QNEC saturation. The first is the aforementioned near vacuum state calculation. The second is a new calculation of the fourth shape variation of *vacuum* entanglement entropy which is also sensitive to the displacement operator defect OPE. In both cases we find that the only new operator that shows up is the spin-3 light ray operator. The outline of the paper is as follows.

- In Section 7.2, we begin by reviewing the basics of the replica trick and the relevant ideas from defect conformal field theory. We review the spectrum of local operators that are induced on the defect, including the infinite family of so-called higher spin displacement operators. We show that, in an interacting theory, these higher spin operators by themselves cannot contribute to the diagonal QNEC. We also present a present a certain conjecture about the nonlocal defect operators.

- In Section 7.3, we discuss how a delta function appears in the OPE of two displacement operators. We focus on a specific defect operator that limits to T_{++} as $n \to 1$. For this defect operator we derive a prediction for the ratio of the $D_+ D_+$ OPE coefficient and its anomalous defect dimension. In Section 7.4, we check this prediction by making use of a modified Ward identity for the defect theory. In Appendix J-K we also explicitly compute the anomalous dimension and the OPE coefficient to confirm this prediction.

- In Section 7.5, we take up the concern that there could be other operators which lead to delta functions even for interacting CFTs. To do this, we compute the defect four point function $\mathcal{F}_n := \langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}_-(y_3)\hat{D}_-(y_4)\rangle$ in the limit $n \to 1$. From this we can read off the spectrum by analyzing the powers of $|y_1 - y_2|$ that appear as $y_1 \to y_2$. We will find that these powers arise from the light-ray OPE of two averaged null energy operators.

- Finally, in Section 7.6, we check our results by explicitly computing the entanglement entropy second variation in near-vacuum states. By using null quantiation for free theories, we show that our results agree with that of [27].

- In Section 7.7, we end with a discussion of our results.

7.2 Replica Trick and the Displacement Operator

In this section, we will review the replica trick and discuss the connection between entanglement entropy and defect operators. This naturally leads to the displacement operator, which will be the key tool for studying (7.1.6).

As outlined in the introduction, the replica trick instructs us to compute the partition function $Z_n/(Z_1)^n = \mathrm{Tr}[\rho_{\mathcal{R}}^n]$, which can be understood as a path integral on a branched manifold $\mathcal{M}_n(\mathcal{R})$, where taking the product of density matrices acts to glue each consecutive sheet together. Using the state operator correspondence, a general state can be represented by the insertion of of a scalar operator in the Euclidean section, so that

$$Z_n = \langle \psi^{\dagger\otimes n}\psi^{\otimes n}\rangle_{\mathcal{M}_n(\mathcal{R})} \tag{7.2.1}$$

where each ψ is inserted on cyclicly consecutive sheets. Alternatively, we can view this $2n$-point correlation function as being computed not on an n-sheeted manifold but on a manifold with trivial topology in the presence of a codimension 2 twist defect operator

$$Z_n = \langle \Sigma_n^0 \psi^{\dagger\otimes n}\psi^{\otimes n}\rangle_{\mathrm{CFT}^{\otimes n}/\mathbb{Z}_n} \equiv \langle \Sigma_n^\psi \rangle \tag{7.2.2}$$

where we have used a compact notation for the twist operator that includes the state operator insertions: $\Sigma_n^\psi \equiv \Sigma_n^0 \psi^{\dagger\otimes n}\psi^{\otimes n}$. It is convenient (and possible) to orbifold the $\mathrm{CFT}^{\otimes n}$ which projects onto states in the singlet of $\mathbb{Z}_n$. This allows us to work with a CFT that for example has only one conserved stress tensor.

We take the defect Σ_n^0 to be associated to a flat cut of a null plane in Minkowski space. We take the metric to be

$$ds^2 = dz d\bar{z} + d\vec{y}^2 \tag{7.2.3}$$

where z and $\bar{z}$ are complexified lightcone coordinates. That is, on the Lorentzian section we have $z = -x^- = x + i\tau$ and $\bar{z} = x^+ = x - i\tau$. Thus, we take the defect to lie at $x^- = X^-(y) = 0$ and $x^+ = X^+(y) = 0$.

For the case of a flat defect, the operator Σ_n^0 breaks the conformal symmetry group down to $SO(2) \times SO(d-1,1)$, with the $SO(2)$ corresponding to the rotations of the plane orthogonal to the defect. This symmetry group suggests that a bulk dimension-d CFT descends to a dimension $d-2$ defect CFT, which describes the excitations of the defect. We can thus use the language of boundary CFTs to analyze this problem. We will only give a cursory overview of this rich subject. For a more thorough review of the topic see [9, 13, 15], and for additional background see [79, 30, 2, 24]. The important aspect for us will be the spectrum of operators that live on the defect.

The spectrum of operators associated to the twist defect was studied in [9]. In that work, techniques were laid out to understand how bulk primary operators induce operators on the defect. This can be quantitatively understood by examining the two-point function of bulk scalar operators in the limit that they both approach the defect. We imagine that as a bulk operator approaches the defect, we can expand in the transverse distance $|z|$ in a bulk to defect OPE so that

$$\lim_{|z|\to 0} \sum_{k=0}^{n-1} \mathcal{O}^{(k)}(z,\bar{z},y)\Sigma_n^0 = z^{-(\Delta_{\mathcal{O}}+\ell_{\mathcal{O}})}\bar{z}^{-(\Delta_{\mathcal{O}}-\ell_{\mathcal{O}})} \sum_j C_{\mathcal{O}}^j z^{(\hat{\Delta}_j+\ell_j)/2}\bar{z}^{(\hat{\Delta}_j-\ell_j)/2}\hat{\mathcal{O}}_j(y)\Sigma_n^0 \tag{7.2.4}$$

where $\Delta_{\mathcal{O}}$ is the dimension of the bulk operator, while $\hat{\Delta}_j$ is the dimension of the jth defect operator $\hat{\mathcal{O}}_j$. Every operator is also now labeled by its spin, ℓ, under the $SO(2)$ rotations $z \to ze^{-i\phi}$. From the defect CFT point of view, the $SO(2)$ spin is an internal symmetry and the ℓ_j's are the defect operators' associated quantum numbers. Notice that the $\mathbb{Z}_n$ symmetry has the effect of projecting out operators of non-integer spin. This is another reason for why the $\mathbb{Z}_n$ orbifolding is needed for treating the theory on the defect as a normal Euclidean CFT.

Equation (7.2.4) suggests an easy way to obtain defect operators in terms of the bulk operators. Consider the lowest dimension defect operator $\hat{\Delta}_\ell$ of a fixed spin ℓ. Then we can extract the defect operator via a residue projection,

$$\hat{\mathcal{O}}_\ell(0)\Sigma_n^0 = \lim_{|z|\to 0} \frac{|z|^{-\hat{\tau}_\ell+\tau_\alpha}}{2\pi i} \oint \frac{dz}{z} z^{-\ell+\ell_\alpha} \sum_{k=0}^{n-1} \mathcal{O}_\alpha^{(k)}(z,|z|^2/z,0)\Sigma_n^0 \tag{7.2.5}$$

where $\hat{\tau}_\ell$ and τ_α are the twists of the defect and bulk operators respectively. Note that these leading twist operators are necessarily defect primaries.

Note that in general, due to the breaking of full conformal symmetry, $\hat{\Delta}_\ell$ will contain an anomalous dimension $\gamma_\ell(n)$. In this paper we will mainly be interested in the defect spectrum near $n=1$ so after analytically continuing in n we can expand $\gamma_\ell(n)$ around $n=1$ as $\gamma(n) = \gamma^{(0)} + \gamma^{(1)}(n-1) + \mathcal{O}((n-1)^2)$. We now give a brief review of the various defect operators discovered in [9].[2]

[2] See [107] for a complementary method for computing the defect spectrum from the bootstrap and an appropriate Lorentzian inversion formula. It would be interesting to derive some of the results presented here in that language.

Operators induced by bulk scalars or spin one primaries

Associated to each bulk scalar ϕ, or spin-one primary V_μ, of dimension Δ_ϕ, Δ_V, the authors of [9] found a family of defect operators of dimension $\hat{\Delta}^{\ell}_{\phi,V} = \Delta_{\phi,V} - J_{\phi,V} + \ell + \gamma^{(1)}_{\phi,V}(n-1) + \mathcal{O}((n-1)^2)$ with $SO(2)$ spin ℓ along with their defect descendants. Here $J_{\phi,V} = 0, 1$ for ϕ and V respectively and importantly $\ell \geq J$. The anomalous dimensions for the operators induced by bulk scalars, γ_ϕ, are given in formula (3.25) of [9]. We will not be concerned with these two families in this paper.

Operators induced by bulk primaries of spin $J \geq 2$

For primary operators of spin $J \geq 2$, the authors of [9] again found a similar family of operators with dimensions $\hat{\Delta}^{\ell}_{J} = \Delta_J - J + \ell + \gamma^{(1)}_{J,\ell}(n-1) + \mathcal{O}((n-1)^2)$ where $\ell \geq J$.

For a primary of spin $J \geq 2$, there are also $J-1$ "new" operators with $SO(2)$ charge $J-1 \geq \ell \geq 1$. These "displacement operators" can be written at integer n as

$$\hat{D}^{J}_{\ell} = i \oint d\bar{z} \frac{\bar{z}^{J-\ell-1}}{|z|^{\gamma_{J,\ell}(n)}} \sum_{k=0}^{n-1} \mathcal{J}^{(k)}_{+\dots+}(|z|^2/\bar{z}, \bar{z}) \tag{7.2.6}$$

where J is the spin of the bulk primary $\mathcal{J}_{+\dots+}$ and $1 \leq \ell \leq J-1$ is the $SO(2)$ spin of the defect operator. The power of $|z|^\gamma$ accounts for the dependence of the defect operator dimension on n.

We will primarily be interested in the spectrum of T_{++} on the defect for which there is only one displacement operator, $\hat{D}_+$. The displacement operator can also be equivalently defined in terms of the diffeomorphism Ward identity in the presence of the defect [15]

$$\nabla^\mu \langle \Sigma^\psi_n T_{\mu\nu} \rangle = \delta(z, \bar{z}) \langle \Sigma^\psi_n \hat{D}_\nu \rangle. \tag{7.2.7}$$

This implies that $\hat{D}_+$ corresponds to a null deformation of the orbifold partition function with respect to the entangling surface. In particular, entropy variations are given by $\hat{D}_+$ insertions in the limit $n \to 1$:

$$\langle \Sigma^\psi_n \hat{D}_+(y) \rangle = (n-1) \langle \Sigma^\psi_n \rangle \frac{\delta S_\psi}{\delta x^+(y)} + \mathcal{O}((n-1)^2) \tag{7.2.8}$$

The generalization to two derivatives is then just

$$\langle \Sigma^\psi_n \hat{D}_+(y) \hat{D}_+(y') \rangle = (n-1) \langle \Sigma^\psi_n \rangle \frac{\delta^2 S_\psi}{\delta X^+(y) X^+(y')} + \mathcal{O}((n-1)^2). \tag{7.2.9}$$

We see importantly that statements about entropy variations can be related directly to displacement operator correlation functions.

7.3 Towards saturation of the QNEC

With the displacement operator in hand, we can now describe an argument for QNEC saturation. As just described, second derivatives of the entanglement entropy can be computed via two point functions of the defect CFT displacement operator. Thus, we are interested in proving the following identity:

$$\lim_{n\to 1}\frac{1}{n-1}\langle \Sigma_n^{\psi}\hat{D}_+(y)\hat{D}_+(y')\rangle = 2\pi\,\langle \hat{T}_{++}(y)\rangle_{\psi}\,\delta^{d-2}(y-y') \\ + (\text{less divergent in } |y-y'|) \tag{7.3.1}$$

where $|\psi\rangle$ is any well-defined state in the CFT.

Since we are only interested in the short distance behavior of this equality - namely the delta function piece - we can examine the OPE of the displacement operators

$$\frac{1}{n-1}\hat{D}_+(y)\hat{D}_+(y') = \frac{1}{n-1}\sum_{\alpha}\frac{c_\alpha(n)\hat{\mathcal{O}}^{\alpha}_{++}(y)}{|y-y'|^{2(d-1)-\Delta_\alpha+\gamma_\alpha(n)}} + \text{descendants} \tag{7.3.2}$$

where Δ_α is the dimension of the defect primary $\hat{\mathcal{O}}_\alpha$ at $n=1$ and $\gamma_\alpha(n)$ gives the n dependence of the dimension away from $n=1$. We will refer to $\gamma_\alpha(n)$ as an anomalous dimension. Note that this is an OPE defined purely in the defect CFT. The $++$ labels denote the $SO(2)$ spin of the defect operator, which must match on both sides of the equation. The dimension of the displacement operators themselves are independent of n and fixed by a Ward identity to be $d-1$.

At first glance, this equation would suggest that there are no delta functions in the OPE, only power law divergences. In computing the entanglement entropy, however, we are interested in the limit as $n\to 1$. In this limit, it is possible for a power law to turn into a delta function as follows:

$$\lim_{n\to 1}\frac{n-1}{|y-y'|^{d-2-\gamma^{(1)}(n-1)}} = \frac{S_{d-3}}{\gamma^{(1)}}\delta^{(d-2)}(y-y') \tag{7.3.3}$$

where $\gamma = \gamma^{(1)}(n-1) + \mathcal{O}((n-1)^2)$ and S_{d-3} in the area of the $d-3$ sphere. Comparison of equations (7.3.3) and (7.3.2) shows that a delta function can "condense" in the $\hat{D}_+\times\hat{D}_+$ OPE only if the OPE coefficient and anomalous dimension obey

$$c_\alpha(n)/\gamma_\alpha(n) \sim (n-1) + \mathcal{O}((n-1)^2) \tag{7.3.4}$$

as n approaches 1.

This is, however, not sufficient for a delta function to appear in (7.3.2) as $n\to 1$. We also need to have

$$\Delta_\alpha = d \tag{7.3.5}$$

at $n = 1$. In other words, the defect operators we are looking for must limit to an operator of $SO(2)$ spin two and dimension d as the defect disappears. Clearly, the $\ell = 2$ operator induced by the bulk stress tensor, $\hat{T}_{++}$, satisfies these conditions. Indeed, the first law of entanglement necessitates the appearance of $\hat{T}_{++}$ in the $\hat{D}_+ \times \hat{D}_+$ OPE with a delta function (see Section 7.4 below).

Our main claim, (7.3.1), is the statement that no other operator can show up in (7.3.2) whose contribution becomes a delta function in the $n \to 1$ limit. In the rest of this section, we enumerate all the possible operators that could appear in the $\hat{D}_+ \times \hat{D}_+$ OPE (7.3.2).

Defect operators induced by low-dimension scalars

If there exists a scalar operator of dimension $\Delta = d - 2$, then the associated defect operator with $SO(2)$ spin $\ell = 2$ will have dimension $\Delta = d$ at leading order in $n - 1$. This possibility was discussed in [106]. The contribution of such an operator was found to drop out of the final quantity $\langle T_{++} \rangle - \frac{1}{2\pi} S''_{++}$ for holographic CFTs. We expect the same thing to happen in general CFTs in the presence of such an operator, so we ignore this possibility.

$\ell = 2$ operators induced by spin one primaries

As discussed earlier, these defect operators have dimension $\hat{\Delta} = \Delta_V + 1 + \mathcal{O}(n - 1)$. We see that for spin one primaries not saturating the unitarity bound, i.e. $\Delta_V > d - 1$, these cannot contribute delta functions. Actually, since these operators exist in the CFT at $n = 1$, we will argue in the next section that the first law of entanglement forces their OPE coefficients to be of order $(n - 1)^2$.

For spin-one primaries saturating the unitarity bound, V_μ is then the current associated to some internal symmetry. The entropy is uncharged under all symmetries, so such operators cannot contribute to $\hat{D}_+ \times \hat{D}_+$.

$\ell = 2$ higher spin displacement operators

The most natural candidate for contributions to the $\hat{D}_+ \times \hat{D}_+$ OPE are the $\ell = 2$ higher spin displacement operators discussed in the previous section. These operators are given by equation (7.2.6).

To show that such operators do not contribute delta functions to $\hat{D}_+ \times \hat{D}_+$, we need to argue that their dimensions $\Delta_n(\ell = 2, J)$ do not limit to d as $n \to 1$. As discussed in the previous section, the dimensions of the higher spin displacement operators are given by

$$\Delta_n(\ell, J) = \Delta_J - J + \ell + \mathcal{O}(n - 1). \tag{7.3.6}$$

The anomalous dimensions have not yet been computed but we expect them to be of order $n - 1$, although we will not need this calculation here. The important point for us will be that in a CFT with a twist gap, the leading order dimension of these operators is

$$\Delta_n(2, J) = \tau_J + 2 + \mathcal{O}(n - 1) > d \tag{7.3.7}$$

assuming the twist of the bulk primaries satisfies $\tau_J > d-2$. Here we are using a result on the convexity of twist on the leading Regge trajectory for all J proven in [41]. We see that the bulk higher spin operators would need to saturate the unitarity bound to contribute a delta function. Furthermore, there could be defect descendants of the form $(\partial_y^i \partial_y^i)^k \hat{D}^J_{++}(y)$. But such operators will necessarily contribute to the OPE with larger, positive powers of $|y-y'|$, hence they cannot produce delta functions.

Nonlocal defect operators

So far we have focused on the individual contribution of local defect operators and by power counting we see that these operators cannot appear in the diagonal QNEC. At fixed n, it is reasonable to conjecture that this list we just provided is complete. However we have not fully concluded that something more exotic does not appear in the OPE. As discussed in the introduction this possibility arises because the $n \to 1$ limit may not commute with the OPE.

Indeed, we will find evidence that something non-standard does appear in the displacement OPE. In Section 7.5 and Section 7.6 we will present some computations of correlation functions of the displacement operator for particular states and entangling surfaces. In these specific cases we will be able to make the analytic continuation to $n \to 1$ before taking the OPE. In both cases, we find that the power laws as $y_1 \to y_2$ are controlled by the dimensions associated to non-local spin-3 light ray operators [101]. In the discussion section we will come back to the possibility that these contributions come from an infinite tower of the local defect operators that we have thus far enumerated. We conjecture that when this tower is appropriately re-summed, we will find these non-standard contributions to the entanglement entropy.

We will refer to these operators as *nonlocal defect operators*, and we further conjecture that a complete list of such operators and dimensions is determined by the nonlocal $J = 3$ lightray operators that appear in the lightray OPE of two averaged null energy operators as studied in [82, 100] for the CFT *without* a defect. In order to give further evidence for this conjecture, in Section 7.5 we will compute the analytic continuation of the spectrum of operators appearing around $n = 1$ in the $\hat{D}_+ \times \hat{D}_+$ OPE by computing a fourth order shape variation of vacuum entanglement. Our answer is consistent with the above conjecture. While this relies on a specific continuation in n (a specific choice of "state dependence") we think this is strong evidence that we have not missed anything.

Before studying this nonlocal contribution further, we return to the local defect contribution where we would like to check that the ratio of $c(n)/\gamma(n)$ for $\hat{T}_{++}$ obeys (7.3.4).

7.4 Contribution of $\hat{T}_{++}$

In this section, we will review the first law argument which fixes the coefficient of the stress tensor defect operator to leading order in $n-1$. We will then use defect methods to demonstrate that the stress tensor does contribute with the correct ratio of $c(n)$ and $\gamma(n)$ to produce a delta function with the right coefficient demanded by the first law. To do this, we will make use of a slightly modified form of the usual diffeomorphism Ward identity in the presence of a twist defect that will compute $c(n)/\gamma(n)$. In Appendices J and K, we also explicitly calculate $c(n)$ and $\gamma(n)$ separately for the stress tensor and show that they agree with the result of this sub-section.

The First Law

A powerful guiding principle for constraining which defect operators can appear in the OPE (7.3.2) is the first law of entanglement entropy. The entanglement entropy $S(\rho) = -\operatorname{Tr}[\rho \log \rho]$, when viewed as the expectation value of the operator $-\log \rho$, is manifestly non-linear in the state. The first law of entanglement says that if one linearizes the von Neumann entropy about a reference density matrix - σ - then the change in the entropy is just equal to the change in the expectation value of the vacuum modular Hamiltonian. Specifically it says that

$$\delta \operatorname{Tr}[\rho \log \rho] = \operatorname{Tr}[\delta\rho \log \sigma] \tag{7.4.1}$$

where $\rho = \sigma + \delta\rho$.

The case we will be interested in here is when σ is taken to be the vacuum density matrix for the Rindler wedge. The first law then tells us that the *only* contributions to $\langle \Sigma_n^\psi \hat{D}_+(y)\hat{D}_+(y')\rangle$ that are linear in the state as $n \to 1$ must come from the shape variations of the vacuum modular Hamiltonian.

The second shape derivative of the Rindler wedge modular Hamiltonian is easy to compute from the form of the vacuum modular Hamiltonian associated to generalized Rindler regions [138, 52, 99, 39]. Defining $\Delta \langle H^\sigma_{\mathcal{R}}\rangle_\psi = -\operatorname{Tr}[\rho_{\mathcal{R}} \log \sigma_{\mathcal{R}}] + \operatorname{Tr}[\sigma_{\mathcal{R}} \log \sigma_{\mathcal{R}}]$ to be the vacuum subtracted modular Hamiltonian for a general region $\mathcal{R}$ bounded by a cut of the $x^- = 0$ null plane, then we have the simple universal formula

$$\frac{\delta^2 \Delta \langle H^\sigma_{\mathcal{R}}\rangle_\psi}{\delta X^+(y)\delta X^+(y')} = \frac{2\pi}{\hbar} \langle T_{++}\rangle_\psi \, \delta^{(d-2)}(y-y'). \tag{7.4.2}$$

This is a simple but powerful constraint on the displacement operator OPE; it tells us that the only operator on the defect which is manifestly linear in the state as $n \to 1$ and appears in $\hat{D}_+ \times \hat{D}_+$ at $n = 1$ is the stress tensor defect operator

$$\hat{T}_{++} = \oint \frac{d\bar{z}}{\bar{z}|z|^{\gamma_n}} \sum_{j=0}^{n-1} T^{(j)}_{++}(|z|^2/\bar{z}, \bar{z}). \tag{7.4.3}$$

Thus, any other operator which appears in the OPE around $n = 1$ must contribute in a manifestly non-linear fashion. Examining the list of local defect operators discussed in Section 7.3 the only operators that are allowed by the above argument, aside from $\hat{T}_{++}$, are the higher spin displacement operators. As shown in [9] the limit $n \to 1$ of the expectation value of these operators give a contribution that is non-linear in the state.

We will return to these state dependent operators in later sections. Now we check that indeed the stress tensor contributes with the correct coefficient.

Using the modified Ward identity

In Appendix H, we prove the following intuitive identity:

$$\int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')\hat{D}_+(y)T_{--}(w,\bar{w},0)\rangle = -\partial_{\bar{w}}\langle \Sigma_n^0 \hat{D}_+(y)T_{--}(w,\bar{w},0)\rangle. \tag{7.4.4}$$

We now show that the identity (7.4.4) allows us to compute the stress tensor contribution to the $\hat{D}_+ \times \hat{D}_+$ OPE, which can be written as:

$$\hat{D}_+(y)\hat{D}_+(y') \supset \frac{c(n)}{|y-y'|^{d-2-\gamma(n)}}\hat{T}_{++}(y) + \dots \tag{7.4.5}$$

where we have focused on the $\hat{T}_{++}$ contribution and the ellipsis stand for the defect descendants of $\hat{T}_{++}$. We are free to ignore other defect primaries since they get projected out by the $T_{--}(w,\bar{w},0)$ insertion in (7.4.4). Of course, since (7.4.4) involves a y integral, one might worry that we are using the OPE outside its radius of convergence. For now, we will follow through with this heuristic computation using the OPE. At the end of this subsection, we will say a few words about why this is justified.

Inserting (7.4.5) into (7.4.4) and ignoring the descendants, we find

$$\int d^{d-2}y' \frac{c(n)}{|y-y'|^{d-2-\gamma(n)}} \langle \Sigma_n^0 \hat{T}_{++}(y)T_{--}(w,\bar{w},0)\rangle = \frac{c(n)}{\gamma(n)} S_{d-3} \langle \Sigma_n^0 \hat{T}_{++}(y)T_{--}(w,\bar{w},0)\rangle \tag{7.4.6}$$

where S_n is the area of the unit n-sphere. We can write $\hat{T}_{++}(y)$ in terms of T_{++} integrated around the defect:

$$\hat{T}_{++}(y) = -\frac{1}{2\pi i}\sum_{k=0}^{n-1}\oint \frac{d\bar{z}}{\bar{z}|z|^{\gamma(n)}} T_{++}^{(k)}(|z|^2/\bar{z},\bar{z},y) \tag{7.4.7}$$

We now take the $n \to 1$ limit of equation (7.4.4). Since the right hand side starts at order $(n-1)$, we see that $c(n)$ must begin at one higher order in $n-1$ than $\gamma(n)$. Generically we expect $\gamma(n)$ to begin at order $n-1$ and in Appendix K we will see that it does. We thus get the relation

$$\frac{c^{(2)}}{\gamma^{(1)}} \langle \Sigma_1^0 \hat{T}_{++}(y)T_{--}(w,\bar{w},0)\rangle = -\partial_n\big|_{n=1}\partial_{\bar{w}} \langle \Sigma_n^0 \hat{D}_+(y)T_{--}(w,\bar{w},0)\rangle \tag{7.4.8}$$

where $c(n) = c^{(1)}(n-1) + c^{(2)}(n-1)^2 + ...$ and $\gamma(n) = \gamma^{(1)}(n-1) + ...$.

At $n = 1$, $\langle \Sigma_1^0 \hat{T}_{++}(y) T_{--}(w, \bar{w}, 0)\rangle$ is just the usual stress tensor 2-point function. Moreover, we can evaluate the right hand side of (7.4.4) at order $(n-1)$ by following the steps leading up to eq. (3.31) of [9]. This leads to

$$\partial_{\bar{w}} \langle \hat{D}_+(y) T_{--}(w, \bar{w}, 0)\rangle\Big|_{|w|\to 0} = i(n-1) \oint d\bar{z}\ \partial_{\bar{w}} \left(\int_0^{-\infty} \frac{d\lambda\ \lambda^2}{(\lambda-1)^2} \frac{c_T y^4}{4(w\bar{w} - w\bar{z}\lambda + y^2)^{d+2}} \right)\Bigg|_{|w|,|z|\to 0}$$
$$= -2\pi(n-1)\frac{c_T}{4} y^{-2d} \tag{7.4.9}$$

We are then left with the following expressions for c_1 and c_2:

$$c^{(2)} = \frac{2\pi\gamma^{(1)}}{S_{d-3}},\ c^{(1)} = 0 \tag{7.4.10}$$

This is exactly what is needed in order to write (7.4.5) near $y = y'$ as $\hat{D}_+(y)\hat{D}_+(y') \supset \delta^{(d-2)}(y - y')\hat{T}_{++}(y)$.

We now comment on the justification for using the $\hat{D}_+ \times \hat{D}_+$ OPE. Since the left hand side of (7.4.4) involves a y integral over the whole defect, one might worry that the we have to integrate outside the radius of convergence for the $\hat{D}_+ \times \hat{D}_+$ OPE. We see, however, that the y integral produces an enhancement in $(n-1)$ only for the T_{++} primary. In particular, this enhancement does not happen for the descendants of T_{++}. This suggests that if we were to plug in the explicit form of the defect-defect-bulk 3 point function into equation (7.4.4) we would have seen that the $(n-1)$ enhancement comes from a region of the y integral where $\hat{D}_+$ and $\hat{D}_+$ approach each other. We could then effectively cap the integral over y so that it only runs over regions where the OPE is convergent and still land on the same answer. As a check of our reasoning, in Appendices J and K, we also compute the $c(n)$ and $\gamma(n)$ coefficients separately and check that they have the correct ratio.

7.5 Higher order variations of vacuum entanglement

In this section, we return to the possibility mentioned in Section 7.3 that something non-standard might appear in the displacement operator OPE. The authors of [9] argued that they had found a complete list of all local defect operators. This leaves open the possibility that the $n \to 1$ limit behaves in such a way that forces us to re-sum an infinite number of defect operators. In this Section and the next, we will find evidence that indeed this does occur. We will also give evidence that we have found a complete list of such nonlocal operators important for the $\hat{D}_+ \times \hat{D}_+$ OPE. In interacting theories with a twist gap this list does not include an operator with the correct dimension and spin that would contribute a delta function and violate saturation.

To get a better handle on what such a re-summed operator might be, we turn to explicitly computing the spectrum of operators in the $\hat{D} \times \hat{D}$ OPE. To do this, we consider the defect four point function

$$\mathcal{F}_n(y_1, y_2, y_3, y_4) = \langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}_-(y_3)\hat{D}_-(y_4)\rangle \,. \tag{7.5.1}$$

We will consider configurations where $|y_1 - y_2| = |y_3 - y_4|$ are small but $|y_1 - y_4|$ is large. With these kinematics, we can use the $\hat{D} \times \hat{D}$ OPE twice and re-write the four point function as a sum over defect two point functions

$$\mathcal{F}_n = \sum_{\mathcal{O},\mathcal{O}'} \frac{c_{++}^{\mathcal{O}}(n) c_{--}^{\mathcal{O}'}(n) \, \langle \Sigma_n^0 \hat{\mathcal{O}}_{++}(y_2) \hat{\mathcal{O}}'_{--}(y_4)\rangle}{|y_1 - y_2|^{2(d-1)+\hat{\Delta}_n^{\mathcal{O}}} |y_3 - y_4|^{2(d-1)+\hat{\Delta}_n^{\mathcal{O}'}}} \tag{7.5.2}$$

where $\mathcal{O}, \mathcal{O}'$ denote the local defect primaries and their descendants appearing in $\hat{D} \times \hat{D}$. We immediately see that by examining the powers of $|y_1 - y_2|$ appearing in $\mathcal{F}_n$, we can read off the spectrum of operators we are after. That is, at least before taking the limit $n \to 1$. We have not attempted to compute the OPE coefficients explicitly for all the local defect operators. This is left as an important open problem that would greatly clarify some of our discussion, but this is beyond the scope of this paper.

If we assume that the $n \to 1$ limit commutes with the OPE limit $y_1 \to y_2$ we can now find a contradiction. To see this contradiction, we can compute $\lim_{n\to 1} \mathcal{F}_n$ in an alternate manner holding y_1, y_2 fixed and compare to (7.5.2). The main result we will find is that the divergences in $|y_1 - y_2|$ appear to arise from defect operators of dimension $\Delta_{J_*} - J_* + 2$ where $J_* = 3$ and Δ_{J_*} is defined by analytically continuing the dimensions in (7.3.6) to odd J (recall that (7.3.6) was only considered for even spins previously.) Generically we do not expect these particular dimensions to appear in the list of operator dimensions of the local defect operators that we enumerated. However we conjecture that by including such operator dimensions we complete the list of possible powers that can appear in the displacement OPE at $n = 1$.

This discussion further suggests that the final non-local defect operator that makes the leading contribution beside T_{++} should be an analytic continuation in spin of the local higher spin displacement operators. We will come back to this possibility in the discussion.

We now turn to computing $\mathcal{F}_n$ without using the defect OPE. In Appendix L, we explicitly do the analytic continuation of $\mathcal{F}_n$, but here we simply state the answer. We find that $\mathcal{F}_n$ takes the form

$$\begin{aligned}\mathcal{F}_n \sim (n-1) \int ds e^{-s} \Big\langle T_{--}(x^+ = 0, x^- = -1, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^+ = 0, x^- = -e^{-s}, y_4)\Big\rangle \\ + \mathcal{O}\left((n-1)^2\right), \end{aligned} \tag{7.5.3}$$

which can also be written as:

$$\mathcal{F}_n \sim (n-1) \frac{\Big\langle \mathcal{E}_-(y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) \mathcal{E}_-(y_4)\Big\rangle}{\mathrm{vol}\, SO(1,1)}. \tag{7.5.4}$$

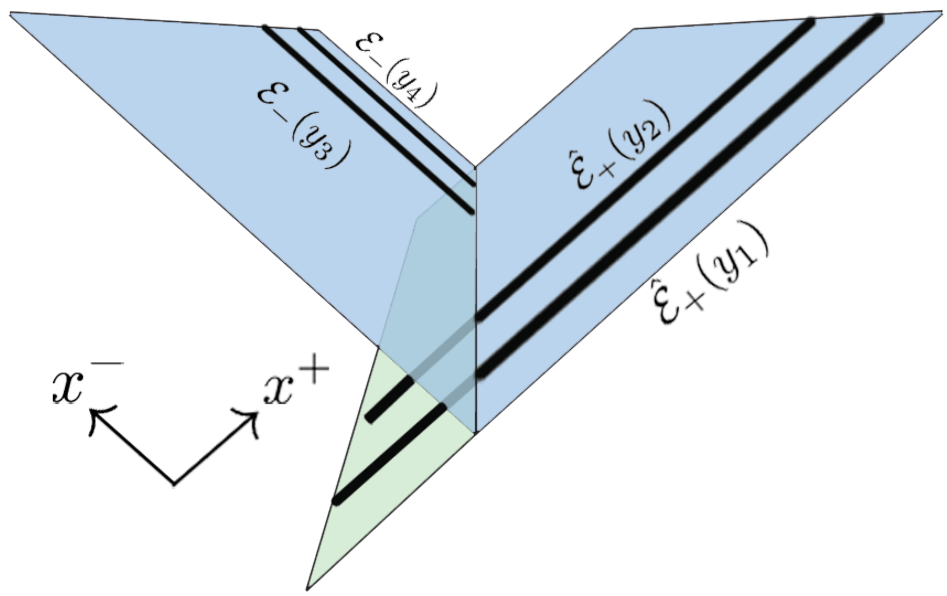

Figure 7.2: The answer for the defect four point function $\mathcal{F}_n$ upon analytic continuation to $n = 1$. We find that there are two insertions of half-averaged null energy operators, $\mathcal{E}_-$, as well as two insertions of $\hat{\mathcal{E}}_+$. Note that strictly speaking, in (7.5.3), the half-averaged null energy operators are inserted in the right Rindler wedge, but by CRT invariance of the vacuum, we can take the half-averaged null energy operators to lie in the left Rindler wedge instead, as in the figure.

The later division by the infinite volume of the 1 dimensional group of boosts is necessary to remove an infinity arising from an overall boost invariance of the four light-ray integrals. See for example [7]. The un-hatted $\mathcal{E}_-$ operators represent half averaged null energy operators, integrated from the entangling surface to infinity. Similar modifications to light-ray operators were used in [100] in order to define their correlation functions and it is necessary here since otherwise the full light-ray operator would annihilate the vacuum.

We see that the effect of two $\hat{D}_+$ insertions was to create two $\hat{\mathcal{E}}_+$ insertions in the limit $n \to 1$. Thus considering the OPE of two displacement operators leads us to the OPE of two null energy operators. This object was studied in [82] and more recently [100]. These authors found that the two averaged null energy insertions can be effectively replaced by a sum over spin 3 "light-ray" operators, one for each Regge trajectory. In other words,

$$\hat{\mathcal{E}}_+(y_1)\hat{\mathcal{E}}_+(y_2) \sim \sum_i \frac{c_i \hat{\mathbb{O}}_i(y_2)}{|y_1 - y_2|^{2(d-2)-\tau^i_{\text{even},J=3}}} \tag{7.5.5}$$

where $\tau^i_{\text{even},J=3}$ is the twist of the even J primaries on the ith Regge trajectory analytically continued down to $J = 3$. A delta function can appear in this expression if $\tau^i_{\text{even},J=3} = d - 2$, i.e. if the dimensions saturate the unitarity bound.

Using the recent results in [41] again, we know that the twists on the leading Regge trajectory obey $\frac{d\tau(J)}{dJ} \geq 0$ and $\frac{d^2\tau(J)}{dJ^2} \leq 0$. Since the stress tensor saturates the unitarity bound, for a theory with a twist gap we know that $\tau^i_{\text{even},J=3} > d-2$, therefore there cannot be a delta function in $y_1 - y_2$. By the previous discussion then, formula (7.5.3) suggests that there are no extra operators besides the stress tensor that produce a delta function. To give further evidence for this we next explicitly work out another case where we can compute the $n \to 1$ limit before we do the OPE and we find the same spectrum of operators.

7.6 Near Vacuum States

We have just seen that the OPE of two displacement operators appears to be controlled by defect operators of dimension $\Delta_{J=3} - 1$. As a check of this result, we will now independently compute the second variation of the entanglement entropy for a special class of states. In these states, we will again see the appearance of the OPE of two null energy operators $\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y')$. This again implies a lack of a delta function for theories with a twist gap.

This computation is particularly illuminating in the case of free field theory where we can use the techniques of null quantization (see Appendix M for a brief review). Null quantization allows us to reduce a computation in a general state of a free theory to a near-vacuum computation. In this way we will also reproduce the computations in [27] using a different method.

The state we will consider is a near vacuum state reduced to a right half-space

$$\rho(\lambda) = \sigma + \lambda\delta\rho + \mathcal{O}(\lambda^2) \tag{7.6.1}$$

where σ is the vacuum reduced to the right Rindler wedge. We can imagine $\rho(\lambda)$ as coming from the following pure state reduced to the right wedge

$$|\psi(\lambda)\rangle = \left(1 + i\lambda \int drd\theta d^{d-2}y g(r,\theta,y)\mathcal{O}(r,\theta,y)\right)|\Omega\rangle + \mathcal{O}(\lambda^2) \tag{7.6.2}$$

where (r,θ,y) are euclidean coordinates centered around the entangling surface and

$$\mathcal{O}(r,\theta,y) = \exp\left(iH_R^\sigma\theta\right)\mathcal{O}(r,0,y)\exp\left(-iH_R^\sigma\theta\right) \tag{7.6.3}$$

where H_R^σ is the Rindler Hamiltonian for the right wedge.

From this expression for $|\Psi(\lambda)\rangle$, we have the formula

$$\delta\rho = \sigma \int drd\theta d^{d-2}y f(r,\theta,y)\mathcal{O}(r,\theta,y) \tag{7.6.4}$$

where

$$f(r,\theta,y) = i\left(g(r,\theta,y) - g(r,2\pi-\theta,y)^*\right). \tag{7.6.5}$$

Note that f obeys the reality condition $f(r,\theta,y) = f(r, 2\pi-\theta,y)^*$.

We are interested in calculating the shape variations of the von-Neumann entropy. To this aim, since the vacuum has trivial shape variations we can compute the vacuum-subtracted entropy ΔS instead. We start by using the following identity

$$\Delta S = \mathrm{Tr}\left((\rho(\lambda) - \sigma)\, H^\sigma \right) - S_{\mathrm{rel}}(\rho(\lambda)|\sigma). \tag{7.6.6}$$

We can now obtain ΔS to second order in λ. The vacuum modular Hamiltonian of the Rindler wedge is just the boost energy

$$\mathrm{Tr}\left[(\rho(\lambda)-\sigma)H^\sigma\right] = \int d^{d-2}y \int dv v\, \mathrm{Tr}\left[\rho(\lambda) T_{++}(u=0,v,y)\right] \tag{7.6.7}$$

where the computation of $S_{\mathrm{rel}}(\rho(\lambda)|\sigma)$ was done in Appendix B of [54]. There it was demonstrated that

$$S_{\mathrm{rel}}(\rho(\lambda)|\sigma) = -\frac{\lambda^2}{2}\int \frac{ds}{4\sinh^2(\frac{s+i\epsilon}{2})}\, \mathrm{Tr}\left[\sigma^{-1}\delta\rho\sigma^{\frac{is}{2\pi}}\delta\rho\sigma^{\frac{-is}{2\pi}}\right] + \mathcal{O}(\lambda^3) \tag{7.6.8}$$

For a pure state like (7.6.2), we can instead write the above expression as a correlation function

$$S_{\mathrm{rel}}(\rho|\sigma) = -\frac{\lambda^2}{2}\int d\mu \int \frac{ds}{4\sinh^2(\frac{s+i\epsilon}{2})} \langle \mathcal{O}(r_1,\theta_1,y_1) e^{is\hat{K}} \mathcal{O}(r_2,\theta_2,y_2)\rangle \tag{7.6.9}$$

where we have used the shorthand

$$\int d\mu = \int dr_{1,2} d\theta_{1,2} d^{d-2}y_{1,2} f(r_1,\theta_1,y_1) f(r_2,\theta_2,y_2) \tag{7.6.10}$$

and $\hat{K} = H^\sigma_R - H^\sigma_L$ is the full modular Hamiltonian associated to Rindler space. This formula (7.6.9) and generalizations has been applied and tested in various contexts [48, 128, 53, 103]. Most of these papers worked with perturbations about a state and a cut with associated to a modular Hamiltonian with a local flow such as the Rindler case. However it turns out that this formula can be applied more widely where $\hat{K}$ need not be local.[3]

We can thus safely replace the Rindler Hamiltonian in (7.6.9) with the Hamiltonian associated to an arbitrary cut of the null plane. This allows us to take shape deformations directly from (7.6.9); by using the algebraic relation for arbitrary-cut modular Hamiltonians [39]

$$e^{-i\hat{K}(X^+)s} e^{i\hat{K}(0)s} = e^{i(e^s-1)\int dy \int dx^+ X^+(y) T_{++}(x^+)} \tag{7.6.11}$$

[3]The only real subtlety is the angular ordering of the insertion of $\mathcal{O}$ in Euclidean. This can be dealt with via an appropriate insertion of the modular conjugation operator - a detail that does not affect the final result. We plan to work out these details in future work.

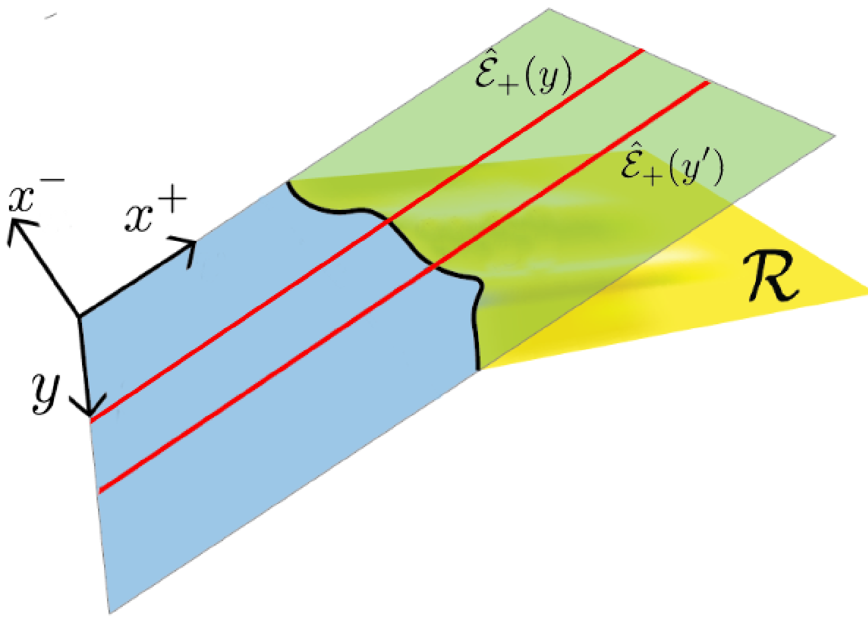

Figure 7.3: For near vacuum states, the insertions of displacement operators limit to two insertions of the averaged null energy operators $\hat{\mathcal{E}}_+$.

we have

$$\frac{\delta^2 S_{\rm rel}(\rho|\sigma)}{\delta X^+(y)\delta X^+(y')} = \frac{\lambda^2}{2}\int d\mu \int ds e^s \langle \mathcal{O}(r_1,\theta_1,y_1)\mathcal{E}_+(y)\mathcal{E}_+(y')e^{is\hat{K}(X^+)}\mathcal{O}(r_2,\theta_2,y_2)\rangle \quad (7.6.12)$$

where the states ρ, σ depend implicitly on $X^+(y)$.[4] Notice that upon taking the variations the double poles in the $1/\sinh^2(s/2)$ kernel of (7.6.8) were precisely canceled by the factors of $e^s - 1$ in the exponent of equation (7.6.11).

This equation is the main result of this section. We see that taking shape derivatives of the entropy can for this class of states be accomplished by insertions of averaged null energy operators. This helps to explain the appearance and disappearance of extra delta functions as we change the coupling in a CFT continuously connected to a free theory. For example, in a free scalar theory, one can show that the OPE contains a delta function,

$$\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y') \supset \delta^{d-2}(y-y'). \quad (7.6.13)$$

This is consistent with the findings of [26] where this extra delta function contribution to the QNEC was computed explicitly. To this aim, in Appendix M, we explicitly reproduce the answer in [26] using the above techniques.

[4] Note the similarity between (7.6.12) and (L.6). This is because one can view the defect four point function in (7.5.3) as going to second order in a state-deformation created by stress tensors with a particular smearing profile.

7.7 Discussion

In this discussion, we briefly elaborate on the possible origin of the non-local operators whose dimensions we found in the displacement operator OPE considered in Sections 7.5 and 7.6. As mentioned in the main text, the appearance of new operators is a bit puzzling since the authors in [9] found a complete set of defect operators as $n \to 1$. In other words, at fixed $n > 1$, it should in principle be possible to expand these new operators as a (perhaps infinite) sum of $\ell = 2$ defect operators.

In particular, we expect them to be representable as an infinite sum over the higher spin displacement operators. We believe that it is necessary to do such an infinite sum before taking the $n \to 1$ limit, which entails that the OPE and replica limits do not commute. This is why [9] did not find such operators. It also seems, given the non-trivial re-derivation of the results in [9] using algebraic tecniques in [40], that these new non-local defect operators are not necessary for the limit $n \to 1$ limit of the bulk to defect OPE used in [9] to compute modular flow correlation functions.

We give the following speculative picture for how the nonlocal defect operators might arise:

$$\hat{D}_+(y_1)\hat{D}_+(y_2) = \frac{c_{J=2}(n)\hat{T}_{++}}{|y_1 - y_2|^{2(d-1)-\Delta_n^{J=2}}} + \sum_{J=3}^{\infty} \frac{c_J(n)\hat{D}_{++}^{(J)}}{|y_1 - y_2|^{2(d-1)-\Delta_n^J}} \tag{7.7.1}$$

where we have suppressed the contribution of defect descendants. The latter sum in (7.7.1) comes from the spin 2 displacement operators that come from the spin J CFT operator. This is a natural infinite class of operators that one could try to re-sum should that prove necessary.

In our calculations, we did not see any powers in $|y_1 - y_2|$ that could be associated to any individual higher spin displacement operator (as in the second term in (7.7.1)). Instead, in Section 7.5 and Section 7.6 after taking the $n \to 1$ limit we observed dimensions that did not belong to any of the known local defect operators. One possibility is that the higher spin operators in (7.7.1) re-sum into a new term that has a non-trivial interplay with the $n \to 1$ limit. One way this might happen is if the OPE coefficients of the higher spin displacement operators take the form

$$c_{J=2k}(n) \sim \frac{1}{(J-3)(n-1)^{J-3}} \tag{7.7.2}$$

so that they diverge as n approaches 1. Such a divergent expansion is highly reminiscent of the Regge limit for four point functions where instead the divergence appears from the choice of kinematics. This pattern of divergence where the degree increases linearly with spin can be handled using the Sommerfeld-Watson trick for re-summing the series. The basic idea is to re-write the sum as a contour integral in the complex J-plane. One then unwraps the contour and picks up various other features depending on the correlator.

Our conjecture in (7.7.2) is that the other features which one encounters upon unwrapping the J contour is quite simple: there is just one pole at $J = 3$. Upon unwrapping the contour in the J-plane, we pick up the pole at $J = 3$, which suggests that indeed these new divergences in $|y_1 - y_2|$ are associated to operators which are analytic continuations in spin of the higher spin displacement operators. In this way we would reproduce the correct power law in $|y_1 - y_2|$ as predicted for near vacuum states.

Note that this needs to be true for *any* CFT - not just at large N or large coupling. The universality of this presumably comes from the universality of three point functions. Indeed, one can try to compute these OPE coefficients. We should consider the following three point function:

$$\langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}^{(J)}_{--}(y_3)\rangle \sim \frac{c_J(n)\,\langle \Sigma_n^0 \hat{D}^{(J)}_{++}(y_2)\hat{D}^{(J)}_{--}(y_3)\rangle}{|y_1 - y_2|^{2(d-1)-\hat{\Delta}_n(J)}} \tag{7.7.3}$$

Via calculations based on the results in Appendix I, we find the three point function above in the the replica limit is:

$$\sim (n-1)\oint dw w^{J-3}\,\langle \mathcal{J}_{-\dots-}(w,\bar{w}=0,y_3)\hat{\mathcal{E}}_+(y_1)\mathcal{E}_+(y_2)\rangle + \mathcal{O}((n-1)^2). \tag{7.7.4}$$

Naively, the full null energy operator $\hat{\mathcal{E}}_+(y_1)$ commutes with the half null energy operator $\mathcal{E}_+(y_2)$ and one can use the fact that $\hat{\mathcal{E}}_+(y_1)\,|\Omega\rangle = 0$ to conclude that $c_J(n=1)$ vanishes. This seems to be incorrect however due to a divergence that arrises in the null energy integrals. Rather we claim that this coefficient diverges. The way to see this is to write

$$\begin{aligned}&\langle \mathcal{J}_{-\dots-}(w,\bar{w}=0,y_3)\hat{\mathcal{E}}_+(y_1)\mathcal{E}_+(y_2)\rangle = \\ &\int_{-\infty}^{\infty} dx_1^+ \int_0^{\infty} dx_2^+\,\langle \mathcal{J}_{-\dots-}(w,\bar{w}=0,y_3)T_{++}(0,x_1^+,y_1)T_{++}(0,x_2^+,y_2)\rangle\,.\end{aligned} \tag{7.7.5}$$

We can now attempt to apply the bulk OPE between the two T_{++}'s which in these kinematics must become[5]

$$T_{++}(x^-=0,x_1^+,y_1)T_{++}(x^-=0,x_2^+,y_2) = \sum_{J=2}^{\infty}\frac{(x_{12}^+)^{J-4}\mathcal{J}^J_{+\dots+}(x_2^+,y_2)}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}} + (\text{descendants}). \tag{7.7.6}$$

where $\hat{\Delta}_1(J) = \Delta_J - J + 2$. Plugging (7.7.6) into (7.7.5) and re-labeling $x_1 \to \lambda_1 x_2$, we see that for even $J \geq 3$, the λ_1 integral has an IR divergence

[5] To get the exact answer, one needs to account for all of the $SO(2)$ descendants in this OPE as well since they contribute equally to the higher spin displacement operator. We expect all of these descendants to have the same scaling behavior with $n-1$ and $J-3$.

One can cut-off the integral over λ_1 at some cutoff Λ. The answer will then diverge like

$$\begin{aligned}
&\frac{\left(\int_{-\Lambda}^{\Lambda} d\lambda_1\, \lambda_1^{J-4}\right)}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}} \times \int_0^\infty dx_2 x_2^{J-3} \left\langle \mathcal{J}_{-\ldots-}(w,\bar{w}=0,y_3)\mathcal{J}_{+\ldots+}(z=0,\bar{z}=x_2^+,y_2)\right\rangle \\
&\sim \frac{\Lambda^{J-3}}{J-3}\int_0^\infty dx_2\, x_2^{J-3} \left\langle \mathcal{J}_{-\ldots-}(w,\bar{w}=0,y_3)\mathcal{J}_{+\ldots+}(z=0,\bar{z}=x_2^+,y_2)\right\rangle \times \frac{1}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}}.
\end{aligned} \tag{7.7.7}$$

The $\mathcal{J}-\mathcal{J}$ correlator on the right is precisely the order $n-1$ piece in $\langle \Sigma_n^0 \hat{D}_{++}^J \hat{D}_{--}^{(J)}\rangle$ so we find that the OPE coefficient scales like $c(n=1) \sim \frac{\Lambda^{J-3}}{J-3}$.

Since Λ is some auxiliary parameter, it is tempting to assign $\Lambda \sim 1/(n-1)$; we then find the conjectured behavior in (7.7.2). This is ad hoc and we do not have an argument for this assignmennt, except to say that the divergence is likely naturally regulated by working at fixed n close to 1. This is technically difficult so we leave this calculation to future work.

Bibliography

[1] Nima Afkhami-Jeddi et al. "Shockwaves from the Operator Product Expansion". In: (Sept. 2017). eprint: 1709.03597. URL: https://arxiv.org/abs/1709.03597.

[2] Cesar Agón and Thomas Faulkner. "Quantum Corrections to Holographic Mutual Information". In: *JHEP* 08 (2016), p. 118. DOI: 10.1007/JHEP08(2016)118. arXiv: 1511.07462 [hep-th].

[3] Chris Akers et al. "Geometric Constraints from Subregion Duality Beyond the Classical Regime". In: (Oct. 2016). eprint: 1610.08968. URL: https://arxiv.org/abs/1610.08968.

[4] Chris Akers et al. "Geometric Constraints from Subregion Duality Beyond the Classical Regime". In: (2016). arXiv: 1610.08968 [hep-th].

[5] Chris Akers et al. "The Quantum Null Energy Condition, Entanglement Wedge Nesting, and Quantum Focusing". In: (June 2017). eprint: 1706.04183. URL: https://arxiv.org/abs/1706.04183.

[6] Andrea Allais and Mark Mezei. "Some results on the shape dependence of entanglement and Renyi entropies". In: *Phys. Rev.* D91.4 (2015), p. 046002. DOI: 10.1103/PhysRevD.91.046002. arXiv: 1407.7249 [hep-th].

[7] Srivatsan Balakrishnan, Souvik Dutta, and Thomas Faulkner. "Gravitational dual of the Renyi twist displacement operator". In: *Phys. Rev.* D96.4 (2017), p. 046019. DOI: 10.1103/PhysRevD.96.046019. arXiv: 1607.06155 [hep-th].

[8] Srivatsan Balakrishnan et al. In: *To Appear* (2018).

[9] Srivatsan Balakrishnan et al. "A General Proof of the Quantum Null Energy Condition". In: (June 2017). eprint: 1706.09432. URL: https://arxiv.org/abs/1706.09432.

[10] Vijay Balasubramanian and Per Kraus. "A Stress Tensor for Anti-de Sitter Gravity". In: (). eprint: hep-th/9902121. URL: https://arxiv.org/abs/hep-th/9902121.

[11] Jacob D. Bekenstein. "Black Holes and Entropy". In: *Phys. Rev.* D7 (1973), pp. 2333–2346. DOI: 10.1103/PhysRevD.7.2333.

[12] Jyotirmoy Bhattacharya et al. "Entanglement Density and Gravitational Thermodynamics". In: *Phys. Rev.* D91.10 (2015), p. 106009. DOI: 10.1103/PhysRevD.91.106009. arXiv: 1412.5472 [hep-th].

[13] Lorenzo Bianchi et al. "Renyi entropy and conformal defects". In: *Journal of High Energy Physics* 2016.7 (2016). ISSN: 1029-8479. DOI: 10.1007/jhep07(2016)076. URL: http://dx.doi.org/10.1007/JHEP07(2016)076.

[14] M. Billó et al. "Line defects in the 3d Ising model". In: *JHEP* 07 (2013), p. 055. DOI: 10.1007/JHEP07(2013)055. arXiv: 1304.4110 [hep-th].

[15] Marco Billo et al. "Defects in conformal field theory". In: *Journal of High Energy Physics* 2016.4 (2016), pp. 1–56. ISSN: 1029-8479. DOI: 10.1007/jhep04(2016)091. URL: http://dx.doi.org/10.1007/JHEP04(2016)091.

[16] J. J Bisognano and E. H. Wichmann. "On the Duality Condition for Quantum Fields". In: *J. Math. Phys.* 17 (1976), pp. 303–321. DOI: 10.1063/1.522898.

[17] Raphael Bousso. "Holography in general space-times". In: *JHEP* 06 (1999), p. 028. DOI: 10.1088/1126-6708/1999/06/028. arXiv: hep-th/9906022 [hep-th].

[18] Raphael Bousso and Netta Engelhardt. "Generalized Second Law for Cosmology". In: *Phys. Rev.* D93.2 (2016), p. 024025. DOI: 10.1103/PhysRevD.93.024025. arXiv: 1510.02099 [hep-th].

[19] Raphael Bousso and Netta Engelhardt. "New Area Law in General Relativity". In: *Phys. Rev. Lett.* 115.8 (2015), p. 081301. DOI: 10.1103/PhysRevLett.115.081301. arXiv: 1504.07627 [hep-th].

[20] Raphael Bousso and Netta Engelhardt. "Proof of a New Area Law in General Relativity". In: *Phys. Rev.* D92.4 (2015), p. 044031. DOI: 10.1103/PhysRevD.92.044031. arXiv: 1504.07660 [gr-qc].

[21] Raphael Bousso et al. "A Quantum Focussing Conjecture". In: (2015). arXiv: 1506.02669 [hep-th].

[22] Raphael Bousso et al. "Entropy on a null surface for interacting quantum field theories and the Bousso bound". In: (June 2014). eprint: 1406.4545. URL: https://arxiv.org/abs/1406.4545.

[23] Raphael Bousso et al. "Entropy on a null surface for interacting quantum field theories and the Bousso bound". In: *Phys. Rev. D* 91 (2015), p. 084030. eprint: 1406.4545. URL: https://arxiv.org/abs/1406.4545.

[24] Raphael Bousso et al. "Entropy on a null surface for interacting quantum field theories and the Bousso bound". In: *Phys. Rev.* D91.8 (2015), p. 084030. DOI: 10.1103/PhysRevD.91.084030. arXiv: 1406.4545 [hep-th].

[25] Raphael Bousso et al. "Proof of a Quantum Bousso Bound". In: *Phys. Rev.* D90.4 (2014), p. 044002. DOI: 10.1103/PhysRevD.90.044002. arXiv: 1404.5635 [hep-th].

[26] Raphael Bousso et al. "Proof of the Quantum Null Energy Condition". In: (2015). arXiv: 1509.02542 [hep-th].

[27] Raphael Bousso et al. "Proof of the Quantum Null Energy Condition". In: *Phys. Rev. D* 93 (2016), p. 024017. eprint: 1509.02542. URL: https://arxiv.org/abs/1509.02542.

[28] William Bunting, Zicao Fu, and Donald Marolf. "A coarse-grained generalized second law for holographic conformal field theories". In: (2015). arXiv: 1509.00074 [hep-th].

[29] Matthias Burkardt. "Light front quantization". In: *Adv. Nucl. Phys.* 23 (1996), pp. 1–74. DOI: 10.1007/0-306-47067-5_1. arXiv: hep-ph/9505259 [hep-ph].

[30] Pasquale Calabrese, John Cardy, and Erik Tonni. "Entanglement entropy of two disjoint intervals in conformal field theory II". In: *J. Stat. Mech.* 1101 (2011), P01021. DOI: 10.1088/1742-5468/2011/01/P01021. arXiv: 1011.5482 [hep-th].

[31] Pasquale Calabrese and John L. Cardy. "Entanglement entropy and quantum field theory". In: *J. Stat. Mech.* 0406 (2004), P06002. DOI: 10.1088/1742-5468/2004/06/P06002. arXiv: hep-th/0405152 [hep-th].

[32] Robert Callan, Jian-Yang He, and Matthew Headrick. "Strong subadditivity and the covariant holographic entanglement entropy formula". In: *JHEP* 06 (2012), p. 081. DOI: 10.1007/JHEP06(2012)081. arXiv: 1204.2309 [hep-th].

[33] Xian O. Camanho et al. "Causality Constraints on Corrections to the Graviton Three-Point Coupling". In: *JHEP* 02 (2016), p. 020. DOI: 10.1007/JHEP02(2016)020. arXiv: 1407.5597 [hep-th].

[34] P. Candelas. "Vacuum polarization in Schwarzschild spacetime". In: *Physical Review D* 21.8 (1980), pp. 2185–2202. DOI: 10.1103/PhysRevD.21.2185.

[35] John Cardy and Erik Tonni. "Entanglement hamiltonians in two-dimensional conformal field theory". In: *J. Stat. Mech.* 1612.12 (2016), p. 123103. DOI: 10.1088/1742-5468/2016/12/123103. arXiv: 1608.01283 [cond-mat.stat-mech].

[36] H. Casini. "Relative entropy and the Bekenstein bound". In: *Class.Quant.Grav.* 25 (2008), p. 205021. eprint: 0804.2182. URL: https://arxiv.org/abs/0804.2182.

[37] Horacio Casini, Marina Huerta, and Robert C. Myers. "Towards a Derivation of Holographic Entanglement Entropy". In: *JHEP* 05 (2011), p. 036. DOI: 10.1007/JHEP05(2011)036. arXiv: 1102.0440 [hep-th].

[38] Horacio Casini, Eduardo Teste, and Gonzalo Torroba. "Markov property of the CFT vacuum and the a-theorem". In: *Phys. Rev. Lett.* 118 (2017), p. 261602. eprint: 1704.01870. URL: https://arxiv.org/abs/1704.01870.

[39] Horacio Casini, Eduardo Teste, and Gonzalo Torroba. "Modular Hamiltonians on the null plane and the Markov property of the vacuum state". In: (Mar. 2017). eprint: 1703.10656. URL: https://arxiv.org/abs/1703.10656.

[40] Fikret Ceyhan and Thomas Faulkner. "Recovering the QNEC from the ANEC". In: (2018). arXiv: 1812.04683 [hep-th].

[41] Miguel S. Costa, Tobias Hansen, and João Penedones. "Bounds for OPE coefficients on the Regge trajectory". In: *Journal of High Energy Physics* 2017.10 (2017). ISSN: 1029-8479. DOI: 10.1007/jhep10(2017)197. URL: http://dx.doi.org/10.1007/JHEP10(2017)197.

[42] Bartlomiej Czech et al. "The Gravity Dual of a Density Matrix". In: *Class. Quant. Grav.* 29 (2012), p. 155009. DOI: 10.1088/0264-9381/29/15/155009. arXiv: 1204.1330 [hep-th].

[43] Xi Dong. "Holographic Entanglement Entropy for General Higher Derivative Gravity". In: *JHEP* 01 (2014), p. 044. DOI: 10.1007/JHEP01(2014)044. arXiv: 1310.5713 [hep-th].

[44] Xi Dong, Daniel Harlow, and Aron C. Wall. "Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality". In: *Phys. Rev. Lett.* 117.2 (2016), p. 021601. DOI: 10.1103/PhysRevLett.117.021601. arXiv: 1601.05416 [hep-th].

[45] Xi Dong and Aitor Lewkowycz. "Entropy, Extremality, Euclidean Variations, and the Equations of Motion". In: (May 2017). eprint: 1705.08453. URL: https://arxiv.org/abs/1705.08453.

[46] Netta Engelhardt and Sebastian Fischetti. "The Gravity Dual of Boundary Causality". In: (2016). arXiv: 1604.03944 [hep-th].

[47] Netta Engelhardt and Aron C. Wall. "Quantum Extremal Surfaces: Holographic Entanglement Entropy Beyond the Classical Regime". In: *JHEP* 01 (2015), p. 073. DOI: 10.1007/JHEP01(2015)073. arXiv: 1408.3203 [hep-th].

[48] Thomas Faulkner. "Bulk Emergence and the RG Flow of Entanglement Entropy". In: (Dec. 2014). eprint: 1412.5648. URL: https://arxiv.org/abs/1412.5648.

[49] Thomas Faulkner, Robert G. Leigh, and Onkar Parrikar. "Shape Dependence of Entanglement Entropy in Conformal Field Theories". In: *JHEP* 1604 (2016), p. 088. eprint: 1511.05179. URL: https://arxiv.org/abs/1511.05179.

[50] Thomas Faulkner, Aitor Lewkowycz, and Juan Maldacena. "Quantum Corrections to Holographic Entanglement Entropy". In: *JHEP* 11 (2013), p. 074. DOI: 10.1007/JHEP11(2013)074. arXiv: 1307.2892 [hep-th].

[51] Thomas Faulkner et al. "Gravitation from Entanglement in Holographic CFTs". In: *JHEP* 03 (2014), p. 051. DOI: 10.1007/JHEP03(2014)051. arXiv: 1312.7856 [hep-th].

[52] Thomas Faulkner et al. "Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition". In: (2016). arXiv: 1605.08072 [hep-th].

[53] Thomas Faulkner et al. "Nonlinear Gravity from Entanglement in Conformal Field Theories". In: *JHEP* 08 (2017), p. 057. DOI: `10.1007/JHEP08(2017)057`. arXiv: `1705.03026 [hep-th]`.

[54] Thomas Faulkner et al. "Nonlinear gravity from entanglement in conformal field theories". In: *Journal of High Energy Physics* 2017.8 (2017). ISSN: 1029-8479. DOI: `10.1007/jhep08(2017)057`. URL: `http://dx.doi.org/10.1007/JHEP08(2017)057`.

[55] C.J. Fewster and T.A. Roman. "Null energy conditions in quantum field theory". In: (). eprint: `gr-qc/0209036`. URL: `https://arxiv.org/abs/gr-qc/0209036`.

[56] L. H. Ford and Thomas A. Roman. "Averaged Energy Conditions and Quantum Inequalities". In: *Phys. Rev.* D51 (1995), pp. 4277–4286. DOI: `10.1103/PhysRevD.51.4277`. arXiv: `gr-qc/9410043 [gr-qc]`.

[57] L. H. Ford and Thomas A. Roman. "Restrictions on Negative Energy Density in Flat Space-Time". In: *Phys. Rev.* D55 (1997), pp. 2082–2089. DOI: `10.1103/PhysRevD.55.2082`. arXiv: `gr-qc/9607003 [gr-qc]`.

[58] L. H. Ford and Thomas A. Roman. "The Quantum Interest Conjecture". In: *Phys. Rev.* D60 (1999), p. 104018. DOI: `10.1103/PhysRevD.60.104018`. arXiv: `gr-qc/9901074 [gr-qc]`.

[59] Ben Freivogel and Dimitrios Krommydas. "Wormhole Wars IV: A New Bound". In: (2018). arXiv: `1807.03808 [hep-th]`.

[60] John L. Friedman, Kristin Schleich, and Donald M. Witt. "Topological Censorship". In: *Phys. Rev. Lett.* 71 (1993). [Erratum: Phys. Rev. Lett.75,1872(1995)], pp. 1486–1489. DOI: `10.1103/PhysRevLett.71.1486`. arXiv: `gr-qc/9305017 [gr-qc]`.

[61] Zicao Fu, Jason Koeller, and Donald Marolf. "Violating the Quantum Focusing Conjecture and Quantum Covariant Entropy Bound in $d \geq 5$ dimensions". In: *Class. Quantum Grav.* 34 (2017), p. 175006. eprint: `1705.03161`. URL: `https://arxiv.org/abs/1705.03161`.

[62] Zicao Fu and Donald Marolf. "A bare quantum null energy condition". In: (Nov. 2017). eprint: `1711.02330`. URL: `https://arxiv.org/abs/1711.02330`.

[63] Zicao Fu and Donald Marolf. "Does Horizon Entropy Satisfy a Quantum Null Energy Conjecture?" In: (2016). arXiv: `1606.04713 [hep-th]`.

[64] Davide Gaiotto, Dalimil Mazac, and Miguel F. Paulos. "Bootstrapping the 3d Ising twist defect". In: *JHEP* 03 (2014), p. 100. DOI: `10.1007/JHEP03(2014)100`. arXiv: `1310.5078 [hep-th]`.

[65] Ping Gao, Daniel Louis Jafferis, and Aron Wall. "Traversable Wormholes via a Double Trace Deformation". In: (2016). arXiv: `1608.05687 [hep-th]`.

[66] Sijie Gao and Robert M. Wald. "Theorems on Gravitational Time Delay and Related Issues". In: *Class. Quant. Grav.* 17 (2000), pp. 4999–5008. DOI: `10.1088/0264-9381/17/24/305`. arXiv: `gr-qc/0007021 [gr-qc]`.

[67] Ferdinando Gliozzi et al. "Boundary and Interface CFTs from the Conformal Bootstrap". In: *JHEP* 05 (2015), p. 036. DOI: 10.1007/JHEP05(2015)036. arXiv: 1502.07217 [hep-th].

[68] C. Robin Graham and Edward Witten. "Conformal Anomaly of Submanifold Observables in AdS / CFT Correspondence". In: *Nucl. Phys.* B546 (1999), pp. 52–64. DOI: 10.1016/S0550-3213(99)00055-3. arXiv: hep-th/9901021 [hep-th].

[69] Noah Graham and Ken D. Olum. "Achronal averaged null energy condition". In: *Phys. Rev.* D76 (2007), p. 064001. DOI: 10.1103/PhysRevD.76.064001. arXiv: 0705.3193 [gr-qc].

[70] Steven S. Gubser. "AdS / CFT and gravity". In: *Phys. Rev.* D63 (2001), p. 084017. DOI: 10.1103/PhysRevD.63.084017. arXiv: hep-th/9912001 [hep-th].

[71] Metin Gurses, Tahsin Cagri Sisman, and Bayram Tekin. "New Exact Solutions of Quadratic Curvature Gravity". In: *Phys. Rev.* D86 (2012), p. 024009. DOI: 10.1103/PhysRevD.86.024009. arXiv: 1204.2215 [hep-th].

[72] Daniel Harlow. "The Ryu-Takayanagi Formula from Quantum Error Correction". In: (2016). arXiv: 1607.03901 [hep-th].

[73] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence". In: *Commun. Math. Phys.* 217 (2001), pp. 595–622. DOI: 10.1007/s002200100381. arXiv: hep-th/0002230 [hep-th].

[74] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. "Holographic Reconstruction of Space-Time and Renormalization in the AdS / CFT Correspondence". In: *Commun. Math. Phys.* 217 (2001), pp. 595–622. DOI: 10.1007/s002200100381. arXiv: hep-th/0002230 [hep-th].

[75] Thomas Hartman, Sandipan Kundu, and Amirhossein Tajdini. "Averaged Null Energy Condition from Causality". In: (Oct. 2016). eprint: 1610.05308. URL: https://arxiv.org/abs/1610.05308.

[76] Thomas Hartman, Sandipan Kundu, and Amirhossein Tajdini. "Averaged Null Energy Condition from Causality". In: (2016). arXiv: 1610.05308 [hep-th].

[77] S. W. Hawking. "Gravitational Radiation from Colliding Black Holes". In: *Phys. Rev. Lett.* 26 (1971), pp. 1344–1346. DOI: 10.1103/PhysRevLett.26.1344.

[78] S. W. Hawking. "The Chronology Protection Conjecture". In: *Phys. Rev.* D46 (1992), pp. 603–611. DOI: 10.1103/PhysRevD.46.603.

[79] Matthew Headrick. "Entanglement Renyi entropies in holographic theories". In: *Phys. Rev.* D82 (2010), p. 126010. DOI: 10.1103/PhysRevD.82.126010. arXiv: 1006.0047 [hep-th].

[80] Matthew Headrick. "General properties of holographic entanglement entropy". In: *JHEP* 03 (2014), p. 085. DOI: 10.1007/JHEP03(2014)085. arXiv: 1312.6717 [hep-th].

[81] Matthew Headrick et al. "Causality & Holographic Entanglement Entropy". In: *JHEP* 12 (2014), p. 162. DOI: 10.1007/JHEP12(2014)162. arXiv: 1408.6300 [hep-th].

[82] Diego M Hofman and Juan Maldacena. "Conformal collider physics: energy and charge correlations". In: *Journal of High Energy Physics* 2008.05 (2008), pp. 012–012. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2008/05/012. URL: http://dx.doi.org/10.1088/1126-6708/2008/05/012.

[83] Christoph Holzhey, Finn Larsen, and Frank Wilczek. "Geometric and renormalized entropy in conformal field theory". In: *Nucl. Phys.* B424 (1994), pp. 443–467. DOI: 10.1016/0550-3213(94)90402-2. arXiv: hep-th/9403108 [hep-th].

[84] Gerard 't Hooft. "Dimensional Reduction in Quantum Gravity". In: *arXiv* (1993).

[85] Veronika E. Hubeny and Mukund Rangamani. "Causal Holographic Information". In: *JHEP* 06 (2012), p. 114. DOI: 10.1007/JHEP06(2012)114. arXiv: 1204.1698 [hep-th].

[86] Veronika E. Hubeny, Mukund Rangamani, and Tadashi Takayanagi. "A Covariant holographic entanglement entropy proposal". In: *JHEP* 07 (2007), p. 062. DOI: 10.1088/1126-6708/2007/07/062. arXiv: 0705.0016 [hep-th].

[87] Ling-Yan Hung, Robert C. Myers, and Michael Smolkin. "Some Calculable Contributions to Holographic Entanglement Entropy". In: *JHEP* 08 (2011), p. 039. DOI: 10.1007/JHEP08(2011)039. arXiv: 1105.6055 [hep-th].

[88] C. Imbimbo et al. "Diffeomorphisms and Holographic Anomalies". In: *Class. Quant. Grav.* 17 (2000), pp. 1129–1138. DOI: 10.1088/0264-9381/17/5/322. arXiv: hep-th/9910267 [hep-th].

[89] Ted Jacobson. "Entanglement Equilibrium and the Einstein Equation". In: *Phys. Rev. Lett.* 116 (2016), p. 201101. eprint: 1505.04753. URL: https://arxiv.org/abs/1505.04753.

[90] Ted Jacobson. "Thermodynamics of Spacetime: The Einstein Equation of State". In: *Phys.Rev.Lett.* 75 (1995), pp. 1260–1263. eprint: gr-qc/9504004. URL: https://arxiv.org/abs/gr-qc/9504004.

[91] Ted Jacobson and Renaud Parentani. "Horizon entropy". In: *Found. Phys.* 33 (2003), pp. 323–348. DOI: 10.1023/A:1023785123428. arXiv: gr-qc/0302099 [gr-qc].

[92] Daniel L. Jafferis et al. "Relative entropy equals bulk relative entropy". In: *JHEP* 06 (2016), p. 004. DOI: 10.1007/JHEP06(2016)004. arXiv: 1512.06431 [hep-th].

[93] William R. Kelly and Aron C. Wall. “Holographic Proof of the Averaged Null Energy Condition”. In: *Phys. Rev.* D90.10 (2014). [Erratum: Phys. Rev.D91,no.6,069902(2015)], p. 106003. DOI: 10.1103/PhysRevD.90.106003,10.1103/PhysRevD.91.069902. arXiv: 1408.3566 [gr-qc].

[94] Igor R. Klebanov and Edward Witten. “AdS / CFT correspondence and symmetry breaking”. In: *Nucl. Phys.* B556 (1999), pp. 89–114. DOI: 10.1016/S0550-3213(99)00387-9. arXiv: hep-th/9905104 [hep-th].

[95] G. Klinkhammer. “Averaged energy conditions for free scalar fields in flat spacetimes”. In: *Phys. Rev.* D43 (1991), pp. 2542–2548. DOI: 10.1103/PhysRevD.43.2542.

[96] Jason Koeller and Stefan Leichenauer. “Holographic Proof of the Quantum Null Energy Condition”. In: (2015). arXiv: 1512.06109 [hep-th].

[97] Jason Koeller and Stefan Leichenauer. “Holographic Proof of the Quantum Null Energy Condition”. In: *Phys. Rev. D* 94 (2016), p. 024026. eprint: 1512.06109. URL: https://arxiv.org/abs/1512.06109.

[98] Jason Koeller et al. “In Progress”. In: ().

[99] Jason Koeller et al. “Local Modular Hamiltonians from the Quantum Null Energy Condition”. In: (Feb. 2017). eprint: 1702.00412. URL: https://arxiv.org/abs/1702.00412.

[100] Murat Kologlu et al. “The light-ray OPE and conformal colliders”. In: (2019). arXiv: 1905.01311 [hep-th].

[101] Petr Kravchuk and David Simmons-Duffin. “Light-ray operators in conformal field theory”. In: *JHEP* 11 (2018), p. 102. DOI: 10.1007/JHEP11(2018)102. arXiv: 1805.00098 [hep-th].

[102] Nima Lashkari. “Modular Hamiltonian for Excited States in Conformal Field Theory”. In: *Phys. Rev. Lett.* 117.4 (2016), p. 041601. DOI: 10.1103/PhysRevLett.117.041601. arXiv: 1508.03506 [hep-th].

[103] Nima Lashkari, Hong Liu, and Srivatsan Rajagopal. “Perturbation Theory for the Logarithm of a Positive Operator”. In: (2018). arXiv: 1811.05619 [hep-th].

[104] Stefan Leichenauer. “The Quantum Focusing Conjecture Has Not Been Violated”. In: (2017). arXiv: 1705.05469 [hep-th].

[105] Stefan Leichenauer. “The Quantum Focusing Conjecture Has Not Been Violated”. In: (May 2017). eprint: 1705.05469. URL: https://arxiv.org/abs/1705.05469.

[106] Stefan Leichenauer, Adam Levine, and Arvin Shahbazi-Moghaddam. “Energy is Entanglement”. In: (2018). arXiv: 1802.02584 [hep-th].

[107] Madalena Lemos et al. “Universality at large transverse spin in defect CFT”. In: *JHEP* 09 (2018), p. 091. DOI: 10.1007/JHEP09(2018)091. arXiv: 1712.08185 [hep-th].

[108] Adam R. Levine. "A Holographic Dual of the Quantum Inequalities". In: (2016). arXiv: `1605.05751 [gr-qc]`.

[109] Aitor Lewkowycz and Juan Maldacena. "Generalized Gravitational Entropy". In: *JHEP* 08 (2013), p. 090. DOI: `10.1007/JHEP08(2013)090`. arXiv: `1304.4926 [hep-th]`.

[110] Elliott H. Lieb and Mary Beth Ruskai. "Proof of the strong subadditivity of quantum-mechanical entropy". In: *Journal of Mathematical Physics* 14.12 (1973), pp. 1938–1941. ISSN: 1089-7658. DOI: `10.1063/1.1666274`. URL: `http://dx.doi.org/10.1063/1.1666274`.

[111] Juan Maldacena, Alexey Milekhin, and Fedor Popov. "Traversable wormholes in four dimensions". In: (2018). arXiv: `1807.04726 [hep-th]`.

[112] Juan Maldacena and Leonard Susskind. "Cool horizons for entangled black holes". In: (June 2013). eprint: `1306.0533`. URL: `https://arxiv.org/abs/1306.0533`.

[113] Juan Martin Maldacena. "The Large N limit of superconformal field theories and supergravity". In: *Int. J. Theor. Phys.* 38 (1999). [Adv. Theor. Math. Phys.2,231(1998)], pp. 1113–1133. DOI: `10.1023/A:1026654312961`. arXiv: `hep-th/9711200 [hep-th]`.

[114] Donald Marolf and Aron C. Wall. "State-Dependent Divergences in the Entanglement Entropy". In: *JHEP* 10 (2016), p. 109. eprint: `1607.01246`. URL: `https://arxiv.org/abs/1607.01246`.

[115] Márk Mezei. "Entanglement entropy across a deformed sphere". In: *Phys. Rev. D* 91 (2015), p. 045038.

[116] Robert C. Myers, Razieh Pourhasan, and Michael Smolkin. "On Spacetime Entanglement". In: *JHEP* 06 (2013), p. 013. DOI: `10.1007/JHEP06(2013)013`. arXiv: `1304.2030 [hep-th]`.

[117] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information: 10th Anniversary Edition*. 10th. New York, NY, USA: Cambridge University Press, 2011. ISBN: 1107002176, 9781107002173.

[118] Masahiro Nozaki et al. "Dynamics of Entanglement Entropy from Einstein Equation". In: (Apr. 2013). eprint: `1304.7100`. URL: `https://arxiv.org/abs/1304.7100`.

[119] Ken D. Olum. "Superluminal Travel Requires Negative Energies". In: *Phys. Rev. Lett.* 81 (1998), pp. 3567–3570. DOI: `10.1103/PhysRevLett.81.3567`. arXiv: `gr-qc/9805003 [gr-qc]`.

[120] Roger Penrose. "Gravitational Collapse and Space-Time Singularities". In: *Phys. Rev. Lett.* 14 (1965), pp. 57–59. DOI: `10.1103/PhysRevLett.14.57`.

[121] Mark Van Raamsdonk. "Building up spacetime with quantum entanglement". In: *Gen.Rel.Grav.42:2323-2329,2010; Int.J.Mod.Phys.D* 19 (2010), pp. 2429–2435. eprint: `1005.3035`. URL: `https://arxiv.org/abs/1005.3035`.

[122] Lisa Randall and Raman Sundrum. "A Large mass hierarchy from a small extra dimension". In: *Phys. Rev. Lett.* 83 (1999), pp. 3370–3373. DOI: 10.1103/PhysRevLett.83.3370. arXiv: hep-ph/9905221 [hep-ph].

[123] Lisa Randall and Raman Sundrum. "An Alternative to compactification". In: *Phys. Rev. Lett.* 83 (1999), pp. 4690–4693. DOI: 10.1103/PhysRevLett.83.4690. arXiv: hep-th/9906064 [hep-th].

[124] Paola Ruggiero and Pasquale Calabrese. "Relative Entanglement Entropies in 1+1-dimensional conformal field theories". In: *JHEP* 02 (2017), p. 039. DOI: 10.1007/JHEP02(2017)039. arXiv: 1612.00659 [hep-th].

[125] Shinsei Ryu and Tadashi Takayanagi. "Aspects of Holographic Entanglement Entropy". In: *JHEP* 08 (2006), p. 045. DOI: 10.1088/1126-6708/2006/08/045. arXiv: hep-th/0605073 [hep-th].

[126] Shinsei Ryu and Tadashi Takayanagi. "Holographic derivation of entanglement entropy from AdS/CFT". In: *Phys. Rev. Lett.* 96 (2006), p. 181602. DOI: 10.1103/PhysRevLett.96.181602. arXiv: hep-th/0603001 [hep-th].

[127] Fabio Sanches and Sean J. Weinberg. "A Holographic Entanglement Entropy Conjecture for General Spacetimes". In: (Mar. 2016). eprint: 1603.05250. URL: https://arxiv.org/abs/1603.05250.

[128] Gábor Sárosi and Tomonori Ugajin. "Modular Hamiltonians of excited states, OPE blocks and emergent bulk fields". In: *JHEP* 01 (2018), p. 012. DOI: 10.1007/JHEP01(2018)012. arXiv: 1705.01486 [hep-th].

[129] Gábor Sárosi and Tomonori Ugajin. "Relative entropy of excited states in two dimensional conformal field theories". In: *JHEP* 07 (2016), p. 114. DOI: 10.1007/JHEP07(2016)114. arXiv: 1603.03057 [hep-th].

[130] A. Schwimmer and S. Theisen. "Entanglement Entropy, Trace Anomalies and Holography". In: *Nucl. Phys.* B801 (2008), pp. 1–24. DOI: 10.1016/j.nuclphysb.2008.04.015. arXiv: 0802.1017 [hep-th].

[131] Geoffrey L. Sewell. "Quantum fields on manifolds: PCT and gravitationally induced thermal states". In: *Annals Phys.* 141 (1982), pp. 201–224. DOI: 10.1016/0003-4916(82)90285-8.

[132] L. Susskind. "The World as a Hologram". In: *J.Math.Phys.* 36 (1995), pp. 6377–6396.

[133] Brian Swingle and Mark Van Raamsdonk. "Universality of Gravity from Entanglement". In: (2014). arXiv: 1405.2933 [hep-th].

[134] Herman L. Verlinde. "Holography and compactification". In: *Nucl. Phys.* B580 (2000), pp. 264–274. DOI: 10.1016/S0550-3213(00)00224-8. arXiv: hep-th/9906182 [hep-th].

[135] Matt Visser. "Gravitational vacuum polarization. 2: Energy conditions in the Boulware vacuum". In: *Phys. Rev.* D54 (1996), pp. 5116–5122. DOI: 10.1103/PhysRevD.54.5116. arXiv: gr-qc/9604008 [gr-qc].

[136] Robert M. Wald. *General Relativity*. Chicago: The University of Chicago Press, 1984.

[137] Robert M. Wald and U. Yurtsever. "General proof of the averaged null energy condition for a massless scalar field in two-dimensional curved space-time". In: *Phys. Rev.* D44 (1991), pp. 403–416. DOI: 10.1103/PhysRevD.44.403.

[138] Aron C. Wall. "A proof of the generalized second law for rapidly changing fields and arbitrary horizon slices". In: *Phys. Rev.* D85 (2012). [Erratum: Phys. Rev.D87,no.6,069904(2013)], p. 104049. DOI: 10.1103/PhysRevD.87.069904,10.1103/PhysRevD.85.104049. arXiv: 1105.3445 [gr-qc].

[139] Aron C. Wall. "A Proof of the Generalized Second Law for Rapidly-Evolving Rindler Horizons". In: *Phys. Rev.* D82 (2010), p. 124019. DOI: 10.1103/PhysRevD.82.124019. arXiv: 1007.1493 [gr-qc].

[140] Aron C. Wall. "Maximin Surfaces, and the Strong Subadditivity of the Covariant Holographic Entanglement Entropy". In: *Class. Quant. Grav.* 31.22 (2014), p. 225007. DOI: 10.1088/0264-9381/31/22/225007. arXiv: 1211.3494 [hep-th].

[141] Aron C. Wall. "Proving the Achronal Averaged Null Energy Condition from the Generalized Second Law". In: *Phys. Rev.* D81 (2010), p. 024038. DOI: 10.1103/PhysRevD.81.024038. arXiv: 0910.5751 [gr-qc].

[142] Aron C. Wall. "The Generalized Second Law Implies a Quantum Singularity Theorem". In: *Class. Quant. Grav.* 30 (2013). [Erratum: Class. Quant. Grav.30,199501(2013)], p. 165003. DOI: 10.1088/0264-9381/30/19/199501,10.1088/0264-9381/30/16/165003. arXiv: 1010.5513 [gr-qc].

Chapter 8

Appendix

A Notation and Definitions

Basic Notation

Notation for basic bulk and boundary quantities

- Bulk indices are $\mu, \nu, \ldots$.
- Boundary indices are $i, j, \ldots$. Then $\mu = (z, i)$.
- We assume a Fefferman–Graham form for the metric: $ds^2 = \frac{L^2}{z^2}(dz^2 + \bar{g}_{ij}dx^i dx^j)$.
- The expansion for $\bar{g}_{ij}(x, z)$ at fixed x is

$$\bar{g}_{ij} = g_{ij}^{(0)} + z^2 g_{ij}^{(2)} + z^4 g_{ij}^{(4)} + \cdots + z^d \log z g_{ij}^{(d,log)} + z^d g_{ij}^{(d)} + \cdots . \tag{A.1}$$

  The coefficients $g_{ij}^{(n)}$ for $n < d$ and $g_{ij}^{(d,log)}$ are determined in terms of $g_{ij}^{(0)}$, while $g_{ij}^{(d)}$ is state-dependent and contains the energy-momentum tensor of the CFT. If d is even, then $g_{ij}^{(d,log)} = 0$. To avoid clutter we will often write $g_{ij}^{(0)}$ simply as g_{ij}. Unless otherwise indicated, i, j indices are raised and lowered by $g_{ij}^{(0)}$.
- We use $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ to denote bulk curvature tensors, and R, R_{ij}, R_{ijmn} to denote boundary curvature tensors.

Notation for extremal surface and entangling surface quantities

- Extremal surface indices are $\alpha, \beta, \ldots$.
- Boundary indices are $a, b, \ldots$. Then $\alpha = (z, a)$.

- The extremal surface is parameterized by functions $\bar{X}^\mu(z, y^a)$. We choose a gauge such that $X^z = z$, and expand the remaining coordinates as

$$\bar{X}^i = X^i_{(0)} + z^2 X^i_{(2)} + z^4 X^i_{(4)} + \cdots + z^d \log z X^i_{(d,log)} + z^d X^i_{(d)} + \cdots . \tag{A.2}$$

  The coefficients $X^i_{(n)}$ for $n < d$ and $X^i_{(d,log)}$ are determined in terms of $X^i_{(0)}$ and $g^{(0)}_{ij}$, while $X^i_{(d)}$ is state-dependent and is related to the renormalized entropy of the CFT region.

- The extremal surface induced metric will be denoted $\bar{h}_{\alpha\beta}$ and gauge-fixed so that $\bar{h}_{za} = 0$.

- The entangling surface induced metric will be denoted h_{ab}.

- Note that we will often want to expand bulk quantities in z at fixed y instead of fixed x. For instance, the bulk metric at fixed y is

$$\begin{aligned}\bar{g}_{ij}(y, z) = \bar{g}_{ij}(\bar{X}(z, y), z) &= \bar{g}_{ij}(X_{(0)}(y) + z^2 X_{(2)}(y) + \cdots , z) \\ &= g^{(0)}_{ij} + z^2 \left(g^{(2)}_{ij} + X^m_{(2)} \partial_m g^{(0)}_{ij} \right) + \cdots \end{aligned} \tag{A.3}$$

  Similar remarks apply for things like Christoffel symbols. The prescription is to always compute the given quantity as a function of x first, the plug in $\bar{X}(y, z)$ and expand in a Taylor series.

Intrinsic and Extrinsic Geometry

Now will introduce several geometric quantities, and their notations, which we will need. First, we define a basis of surface tangent vectors by

$$e^i_a = \partial_a X^i. \tag{A.4}$$

We will also make use of the convention that ambient tensors which are not inherently defined on the surface but are written with surface indices (a, b, etc.) are defined by contracting with e^i_a. For instance:

$$g^{(2)}_{aj} = e^i_a g^{(2)}_{ij}. \tag{A.5}$$

We can form the surface projector by contracting the surface indices on two copies of e^i_a:

$$P^{ij} = h^{ab} e^i_a e^j_b = e^i_a e^{ja}. \tag{A.6}$$

We introduces a surface covariant derivative D_a that acts as the covariant derivative on both surface and ambient indices. So it is compatible with both metrics:

$$D_a h_{bc} = 0 = D_a g_{ij}. \tag{A.7}$$

Note also that when acting on objects with only ambient indices, we have the relationship

$$D_a V^{ij\cdots}_{pq\cdots} = e^m_a \nabla_m V^{ij\cdots}_{pq\cdots}, \tag{A.8}$$

where ∇_i is the ambient covariant derivative compatible with g_{ij}.

The extrinsic curvature is computed by taking the D_a derivative of a surface basis vector:

$$K^i_{ab} = -D_a e^i_b = -\partial_a e^i_b + \gamma^c_{ab} e^i_b - \Gamma^i_{ab}. \tag{A.9}$$

Note the overall sign we have chosen. Here γ^c_{ab} is the Christoffel symbol of the metric h_{ab}, and the lower indices on the Γ symbol were contracted with two basis tangent vectors to turn them into surface indices. Note that K^i_{ab} is symmetric in its lower indices. It is an exercise to check that it is normal to the surface in its upper index:

$$e_{ic} K^i_{ab} = 0. \tag{A.10}$$

The trace of the extrinsic curvature is denoted by K^i:

$$K^i = h^{ab} K^i_{ab}. \tag{A.11}$$

Below we will introduce the null basis of normal vectors k^i and l^i. Then we can define expansion $\theta_{(k)}$ ($\theta_{(l)}$) and shear $\sigma^{(k)}_{ab}$ ($\sigma^{(l)}_{ab}$) as the trace and traceless parts of $k_i K^i_{ab}$ ($l_i K^i_{ab}$), respectively.

There are a couple of important formulas involving the extrinsic curvature. First is the Codazzi Equation, which can be computed from the commutator of covariant derivatives:

$$\begin{aligned} D_c K^i_{ab} - D_b K^i_{ac} &= (D_b D_c - D_c D_b) e^i_a \\ &= R^i{}_{abc} - r^d{}_{abc} e^i_d. \end{aligned} \tag{A.12}$$

Here $R^i{}_{abc}$ is the ambient curvature (appropriately contracted with surface basis vectors), while $r^d{}_{abc}$ is the surface curvature. We can take traces of this equation to get others. Another useful thing to do is contract this equation with e^i_d and differentiate by parts, which yields the Gauss–Codazzi equation:

$$K_{cdi} K^i_{ab} - K_{bdi} K^i_{ac} = R_{dabc} - r_{dabc}. \tag{A.13}$$

Various traces of this equation are also useful.

Null Normals k and l

A primary object in our analysis is the bull vector k^i, which is orthogonal to the entangling surface and gives the direction of the surface deformation. It will be convenient to also introduce the null normal l^i, which is defined so that $l_i k^i = +1$. This choice of sign is different from the one that is usually made in these sorts of analysis, but it is necessary to avoid a proliferation of minus signs. With this convention, the projector onto the normal space of the surface is

$$N^{ij} \equiv g^{ij} - P^{ij} = k^i l^j + k^j l^i = 2k^{(i}l^{j)}. \tag{A.14}$$

As we did with the tangent vectors e^i_a, we will introduce a shorthand notation to denote contraction with k^i or l^i: any tensor with k or l index means it has been contracted with k^i or l^i. As such we will avoid using the letters k and l as dummy indices. For instance.

$$R_{kl} \equiv k^i l^j R_{ij}. \tag{A.15}$$

Another quantity associated with k^i and l^i is the normal connection w^a, defined through

$$w_a \equiv l_i D_a k^i. \tag{A.16}$$

With this definition, the tangent derivative of k^i can be shown to be

$$D_a k^i = w_a k^i + K^k_{ab} e^{bi}, \tag{A.17}$$

which is a formula that is used repeatedly in our analysis.

At certain intermediate stages of our calculations it will be convenient to define extensions of k^i and l^i off of the entangling surface, so here we will define such an extension. Surface deformations in both the QNEC and QFC follow geodesics generated by k^i, so it makes sense to define k^i to satisfy the geodesic equation:

$$\nabla_k k^i = 0. \tag{A.18}$$

However, we will *not* define l^i by parallel transport along k^i. It is conceptually cleaner to maintain the orthogonality of l^i to the surface even as the surface is deformed along the geodesics generated by k^i. This means that l^i satisfies the equation

$$\nabla_k l^i = -w^a e^i_a. \tag{A.19}$$

These equations are enough to specify l^i and k^i on the null surface formed by the geodesics generated by k^i. To extend k^i and l^i off of this surface, we specify that they are both parallel-transported along l^i. In other words, the null surface generated by k^i forms the initial condition surface for the vector fields k^i and l^i which satisfy the differential equations

$$\nabla_l k^i = 0, \qquad \nabla_l l^i = 0\ . \tag{A.20}$$

This suffices to specify k^i an l^i completely in a neighborhood of the original entangling surface. Now that we have done that, we record the commutator of the two fields for future use:

$$[k,l]^i = \nabla_k l^i - \nabla_l k^i = -w^c e^i_c. \tag{A.21}$$

B Surface Variations

Most of the technical parts of our analysis have to do with variations of surface quantities under the deformation $X^i \to X^i + \delta X^i$ of the surface embedding coordinates. Here δX^i should be interpreted a vector field defined on the surface. In principle it can include both normal and tangential components, but since tangential components do not actually correspond to physical deformations of the surface we will assume that δX^i is normal. The operator δ denotes the change in a quantity under the variation. In the case where $\delta X^i = \partial_\lambda X^i$, which is the case we are primarily interested in, δ can be identified with ∂_λ. With this in mind, we will always impose the geodesic equation on k^i whenever convenient. In terms of the notation we are introducing here, this is

$$\delta k^i = -\Gamma^i_{kk}. \tag{B.1}$$

To make contact with the main text, we will use the notation $k^i \equiv \delta X^i$, and assume that k^i is null since that is ultimately the case we care about. Some of the formulas we discuss below will not depend on the fact that k^i is null, but we will not make an attempt to distinguish them.

Ambient Quantities For ambient quantities, like curvature tensors, the variation δ can be interpreted straightforwardly as $k^i \partial_i$ with no other qualification. Thus we can freely use, for instance, the ambient covariant derivative ∇_k to simplify the calculations of these quantities. Note that δ itself is not the covariant derivative. As defined, δ is a coordinate dependent operator. This may be less-than-optimal from a geometric point of view, but it has the most conceptually straightforward interpretation in terms of the calculus of variations. In all of the variational formulas below, then, we will see explicit Christoffel symbols appear. Of course, ultimately these non-covariant terms must cancel out of physical quantities. That they do serves as a nice check on our algebra.

Tangent Vectors The most fundamental formula is that of the variation of the tangent vectors $e^i_a \equiv \partial_a X^i$. Directly from the definition, we have

$$\delta e^i_a = \partial_a k^i = D_a k^i - \Gamma^i_{ak} = w_a k^i + K^k_{ab} e^{bi} - \Gamma^i_{ak}. \tag{B.2}$$

This formula, together with the discussion of how ambient quantities transform, can be used together to compute the variations of many other quantities.

Intrinsic Geometry and Normal Vectors The intrinsic metric variation is easily computed from the above formula as

$$\delta h_{ab} = 2K^k_{ab}. \tag{B.3}$$

From here we can find the variation of the tangent projector, for instance:

$$\begin{aligned} \delta P^{ij} &= \delta h^{ab} e_a^i e_b^j + 2h^{ab} e_a^{(i} \partial_b k^{j)} \\ &= -2K_k^{ab} e_a^i e_b^j + 2h^{ab} e_a^{(i} D_b k^{j)} - 2h^{ab} e_a^{(i} \Gamma^{j)}_{bk} \\ &= 2w^a e_a^{(i} k^{j)} - 2h^{ab} e_a^{(i} \Gamma^{j)}_{bk}. \end{aligned} \tag{B.4}$$

Notice that the second line features a derivative of $k^i = \delta X^i$. In a context where we are taking functional derivatives, such as when computing equations of motion, this term would require integration by parts. We can write the last line covariantly as

$$\nabla_k P^{ij} = 2w^a e_a^{(i} k^{j)}. \tag{B.5}$$

Earlier we saw that l^i satisfied the equation $\nabla_k l^i = -w^a e_a^i$ as a result of keeping l^i orthogonal to the surface even as the surface is deformed. In the language of this section, this is seen by the following manipulation:

$$e_a^i \delta l_i = -l_i \partial_a k^i = -w_a - \Gamma^l_{ak}. \tag{B.6}$$

Again, note the derivative of k^i. It is easy to confirm that represents the only nonzero component of $\nabla_k l^i$.

The normal connection $w_a = l^i D_a k_i$ makes frequent appearances in our calculations, and we will need to know its variation. We can calculate that as follows:

$$\begin{aligned} \delta w_a &= \delta l^i D_a k_i + l^i \partial_a \delta k_i - l^i \delta \Gamma^n_{ji} e_a^j k_n - l^i \Gamma^n_{ji} \partial_a k^j k_n - l^i \Gamma^n_{ji} e_a^j \delta k_n \\ &= \nabla_k l^i D_a k_i + R_{klak} \\ &= -w^c K_{ac} + R_{klak}. \end{aligned} \tag{B.7}$$

Extrinsic Curvatures The simplest extrinsic curvature variation is that of the trace of the extrinsic curvature

$$\delta K^i = -K^m \Gamma^i_{mk} - D_a D^a k^i - R^i_{mkj} P^{mj} + \left(2D^a(K^k_{ad}) - D_d(K^k)\right) e^{di} - 2K_k^{ab} K^i_{ab} \tag{B.8}$$

Note that the combination $\delta K^i + K^k \Gamma^i_{km} k^m$ is covariant, so it makes sense to write

$$\nabla_k K^i = -D_a D^a k^i - R^i_{mkj} P^{mj} + \left(2D^a(K^k_{ad}) - D_d(K^k)\right) e^{di} - 2K_k^{ab} K^i_{ab} \tag{B.9}$$

This formula is noteworthy because of the first term, which features derivatives of $k^i = \delta X^i$. This is important because when K^i occurs inside of an integral and we want to compute the functional derivative then we have to first integrate by parts to move those derivatives off of k^i. This issue arises when computing Θ as in the QFC, for instance.

We can contract the previous formulas with l^i and k^i to produce other useful formulas. For instance, contracting with k^i leads to

$$\delta K^k = -K^{kab} K^k_{ab} - R_{kk}, \tag{B.10}$$

which is nothing but the Raychaudhuri equation.

The variation of the full extrinsic curvature K^i_{ab} is quite complicated, but we will not needed. However, its contraction with k^i will be useful and so we record it here:

$$k_i\delta K^i_{ab} = -K^j_{ab}\Gamma^m_{jn}k_m k^n - k_i D_a D_b k^i - R_{kakb}. \tag{B.11}$$

C z-Expansions

Bulk Metric

We are focusing on bulk theories with gravitational Lagrangians

$$\mathcal{L} = \frac{1}{16\pi G_N}\left(\frac{d(d-1)}{\tilde{L}^2} + \mathcal{R} + \ell^2\lambda_1\mathcal{R}^2 + \ell^2\lambda_2\mathcal{R}^2_{\mu\nu} + \ell^2\lambda_{GB}\mathcal{L}_{GB}\right). \tag{C.1}$$

where $\mathcal{L}_{GB} = \mathcal{R}^2_{\mu\nu\rho\sigma} - 4\mathcal{R}^2_{\mu\nu} + \mathcal{R}^2$ is the Gauss-Bonnet Lagrangian, ℓ is the cutoff length scale of the bulk effective field theory, and the couplings λ_1, λ_2, and λ_{GB} are defined to be dimensionless. We have decided to include $\mathcal{L}_{GB}$ as part of our basis of interactions rather than $\mathcal{R}^2_{\mu\nu\rho\sigma}$ because of certain nice properties that the Gauss-Bonnet term has, but this is not important.

We recall that the Fefferman–Graham form of the metric is defined by

$$ds^2 = \frac{1}{z^2}(dz^2 + \bar{g}_{ij}dx^i dx^j), \tag{C.2}$$

where $\bar{g}_{ij}(x,z)$ is expanded as a series in z:

$$\bar{g}_{ij} = g^{(0)}_{ij} + z^2 g^{(2)}_{ij} + z^4 g^{(4)}_{ij} + \cdots + z^d \log z g^{(d,log)}_{ij} + z^d g^{(d)}_{ij} + \cdots. \tag{C.3}$$

In principle, one would evaluate the equation of motion from the above Lagrangian using the Fefferman–Graham metric form as an ansatz to compute these coefficients. The results of this calculation are largely in the literature, and we quote them here. To save notational clutter, in this section we will set $g_{ij} = g^{(0)}_{ij}$.

The first nontrivial term in the metric expansion is independent of the higher-derivative couplings, and in fact is completely determined by symmetry [88]:

$$g^{(2)}_{ij} = -\frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)}Rg_{ij}\right). \tag{C.4}$$

The next term is also largely determined by symmetry, except for a pair of coefficients [88]. We are only interested in the kk-component of $g^{(4)}_{ij}$, and where one of the coefficients drops out. The result is

$$\begin{aligned} g^{(4)}_{kk} = \frac{1}{d-4}\Bigg[&\kappa C_{kijm}C_k^{\ ijm} + \frac{1}{8(d-1)}\nabla^2_k R - \frac{1}{4(d-2)}k^i k^j \Box R_{ij} \\ &- \frac{1}{2(d-2)}R^{ij}R_{kikj} + \frac{d-4}{2(d-2)^2}R_{ki}R^i_k + \frac{1}{(d-1)(d-2)^2}RR_{kk}\Bigg], \end{aligned} \tag{C.5}$$

where C_{ijmn} is the Weyl tensor and

$$\kappa = -\lambda_{GB}\frac{\ell^2}{L^2}\left(1+O\left(\frac{\ell^2}{L^2}\right)\right). \tag{C.6}$$

In $d=4$ we will need an expression for $g_{kk}^{(4,log)}$ as well. One can check that this is obtainable from $g_{kk}^{(4)}$ by first multiplying by $4-d$ and then setting $d\to 4$. We record the answer for future reference:

$$g_{kk}^{(4,log)} = -\left[\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{24}\nabla_k^2 R - \frac{1}{8}k^i k^j \Box R_{ij} - \frac{1}{4}R^{ij}R_{kikj} + \frac{1}{12}RR_{kk}\right]. \tag{C.7}$$

Extremal Surface Coordinates

The extremal surface position is determined by extremizing the generalized entropy functional [47, 45]:

$$S_{gen} = \frac{1}{4G_N}\int\sqrt{\bar{h}}\left[1+2\lambda_1\ell^2\mathcal{R} + \lambda_2\ell^2\left(\mathcal{R}_{\mu\nu}\mathcal{N}^{\mu\nu} - \frac{1}{2}\mathcal{K}_\mu\mathcal{K}^\mu\right) + 2\lambda_{GB}\ell^2\bar{r}\right] + S_{bulk}. \tag{C.8}$$

Here we are using $\mathcal{K}^i$ to denote the extrinsic curvature and $\bar{r}$ the intrinsic Ricci scalar of the surface.

The equation of motion comes from varying S_{gen} and is (ignoring the S_{bulk} term for simplicity)

$$\begin{aligned}0 &= \mathcal{K}^\mu\left[1+2\lambda_1\ell^2\mathcal{R} + \lambda_2\ell^2\left(\mathcal{R}_{\rho\nu}\mathcal{N}^{\rho\nu} - \frac{1}{2}\mathcal{K}_\rho\mathcal{K}^\rho\right) + 2\lambda_{GB}\ell^2\bar{r}\right] + 2\lambda_1\ell^2\nabla^\mu\mathcal{R}\\ &+ \lambda_2\ell^2\Big(\mathcal{N}^{\rho\nu}\nabla^\mu\mathcal{R}_{\rho\nu} + 2\mathcal{P}^{\rho\nu}\nabla_\rho\mathcal{R}^\mu_\nu - 2\mathcal{R}^\mu_\rho\mathcal{K}^\rho + 2\mathcal{K}^{\mu\alpha\beta}\mathcal{R}_{\alpha\beta} + D_\alpha D^\alpha\mathcal{K}^\mu\\ &+ \mathcal{K}^\rho\mathcal{R}_{\mu\sigma\rho\nu}\mathcal{P}^{\nu\sigma} + 2\mathcal{K}^{\mu\alpha\beta}\mathcal{K}_\nu\mathcal{K}^\nu_{\alpha\beta}\Big) - 4\lambda_{GB}\ell^2\bar{r}^{\alpha\beta}\mathcal{K}^\mu_{\alpha\beta}.\end{aligned} \tag{C.9}$$

This equation is very complicated, but since we are working in $d\le 5$ dimensions we only need to solve perturbatively in z for $X^i_{(2)}$ and $X^i_{(4)}$[1]. Furthermore, $X^i_{(2)}$ is fully determined by symmetry to be [130]

$$X^i_{(2)} = \frac{1}{2(d-2)}D^a\partial_a X^i_{(0)} = -\frac{1}{2(d-2)}K^i, \tag{C.10}$$

where K^i denotes the extrinsic curvature of the $X^i_{(0)}$ surface, but we are leaving off the (0) in our notation to save space.

[1] It goes without saying that these formulas are only valid for $d>2$ and $d>4$, respectively.

The computation of $X^i_{(4)}$ is straightforward but tedious. We will only need to know $k_i X^i_{(4)}$ (where indices are being raised and lowered with $g^{(0)}_{ij}$), and the answer turns out to be

$$\begin{aligned}
4(d-4)X^k_{(4)} &= 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) \\
&+ K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)} \\
&+ k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm} \\
&+ 8\kappa\sigma^{ab}_{(k)}C_{kalb} - 2(d-4)\Gamma^k_{jm}X^j_{(2)}X^m_{(2)}.
\end{aligned} \tag{C.11}$$

Here κ depends on λ_{GB} as in (C.6). Notice that the last term in this expression is the only source of noncovariant-ness. One can confirm that this noncovariant piece is required from the definition of $X^i_{(4)}$—despite its index, $X^i_{(4)}$ does not transform like a vector under boundary diffeomorphisms.

We also note that the terms in $X^k_{(4)}$ with covariant derivatives of $g^{(2)}_{ij}$ can be simplified using the extended k^i and l^i fields described §A and the Bianchi identity:

$$k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} = -\frac{1}{4(d-1)}\nabla_k R + \frac{1}{d-2}\nabla_l R_{kk}. \tag{C.12}$$

Finally, we record here the formula for $X^k_{(4,log)}$ which is obtained from $X^k_{(4)}$ by multiplying by $4-d$ and sending $d \to 4$:

$$\begin{aligned}
-4X^k_{(4,log)} &= 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) \\
&+ K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)} \\
&+ k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm} \\
&+ 8\kappa\sigma^{ab}_{(k)}C_{kalb}.
\end{aligned} \tag{C.13}$$

We will not bother unpacking all of the definitions, but the main things to notice is that the noncovariant part disappears.

D Details of the EWN Calculations

In this section we provide some insight into the algebra necessary to complete the calculations of the main text, primarily regarding the calculation of the subleading part of $(\delta\bar{X})^2$ in §5.1. The task is to simplify (5.1.13),

$$\begin{aligned}
L^{-2}(\delta\bar{X})^2\big|_{z^2} &= 2k_i\delta X^i_{(4)} + 2g^{(2)}_{ij}k^i\delta X^j_{(2)} + g_{ij}\delta X^i_{(2)}\delta X^j_{(2)} + g^{(4)}_{ij}k^ik^j + X^m_{(4)}\partial_m g_{ij}k^ik^j \\
&+ 2X^m_{(2)}\partial_m g_{ij}k^i\delta X^j_{(2)} + X^m_{(2)}\partial_m g^{(2)}_{ij}k^ik^j + \frac{1}{2}X^m_{(2)}X^n_{(2)}\partial_m\partial_n g_{ij}k^ik^j.
\end{aligned} \tag{D.1}$$

After some algebra, we can write this as

$$L^{-2}(\delta\bar{X})^2\big|_{z^2} = g^{(4)}_{kk} + 2\delta(X^k_{(4,cov)}) + 2g^{(2)}_{ik}\nabla_k X^i_{(2)} + \nabla_k X^{(2)}_j \nabla_k X^j_{(2)} - \frac{1}{d-2}(X^l_{(2)})\nabla_k R_{kk}. \quad \text{(D.2)}$$

Here we have defined

$$X^i_{(4,cov)} = X^i_{(4)} + \frac{1}{2}\Gamma^i_{lm} X^l_{(2)} X^m_{(2)}, \quad \text{(D.3)}$$

which transforms like a vector (unlike $X^i_{(4)}$). From here, the algebra leading to (5.1.14) is mostly straightforward, though tedious. The two main tasks which require further explanation are the simplification of one of the terms in $g^{(4)}_{kk}$ and one of the terms in $\delta X^k_{(4,cov)}$. We will explain those now.

$g^{(4)}_{kk}$ **Simplification** We recall the formula for $g^{(4)}_{kk}$ from (C.5):

$$\begin{aligned} g^{(4)}_{kk} = \frac{1}{d-4}\Bigg[& \kappa C_{kijm} C_k^{\ ijm} + \frac{1}{8(d-1)}\nabla^2_k R - \frac{1}{4(d-2)} k^i k^j \Box R_{ij} \\ & - \frac{1}{2(d-2)} R^{ij} R_{kikj} + \frac{d-4}{2(d-2)^2} R_{ki} R^i_k + \frac{1}{(d-1)(d-2)^2} R R_{kk} \Bigg]. \end{aligned} \quad \text{(D.4)}$$

The main difficulty is with the term $k^i k^j \Box R_{ij}$. We will rewrite this term by making use of the geometric quantities introduced in the other appendices, and in particular we make use of the extended k and l field from §A. We first separate it into two terms:

$$k^i k^j \Box R_{ij} = k^i k^j N^{rs} \nabla_r \nabla_s R_{ij} + k^i k^j P^{rs} \nabla_r \nabla_s R_{ij}. \quad \text{(D.5)}$$

Now we compute each of these terms individually:

$$\begin{aligned} k^i k^j N^{rs} \nabla_r \nabla_s R_{ij} &= 2k^i k^j l^s \nabla_k \nabla_s R_{ij} + 2R_{kmlk} R^m_k \\ &= 2\nabla_k \nabla_l R_{kk} + 2w^c k^i k^j D_c R_{ij} + 2R_{kmlk} R^m_k \\ &= 2\nabla_k \nabla_l R_{kk} + 2w^c D_c R_{kk} - 4w^c w_c R_{kk} - 4w^c K^a_{ck} R_{ka} + 2R_{kmlk} R^m_k \\ &= 2\nabla_k \nabla_l R_{kk} + 2w^c D_c R_{kk} - 4w^c w_c R_{kk} + 2R_{kmlk} R^m_k. \end{aligned} \quad \text{(D.6)}$$

In the last line we assumed that $\sigma_{(k)} = 0$ and $\theta_{(k)} = 0$, which is the only case we will need to worry about. The other term is slightly messier, becoming

$$\begin{aligned} k^i k^j P^{rs} \nabla_r \nabla_s R_{ij} &= k^i k^j e^{sc} D_c \nabla_s R_{ij} \\ &= D_c(k^i k^j D^c R_{ij}) - D_c(k^i k^j e^{sc}) \nabla_s R_{ij} \\ &= D_c(k^i k^j D^c R_{ij}) - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + 6w_c K^{ca}_k R_{ak} \\ &\quad - 2K^{ca}_k D_c R_{ka} + 2K^{ca}_k K^i_{ca} R_{ik} + 2K^{ca}_k K^{bk}_c R_{ab} + K^s \nabla_s R_{kk} \\ &= D_c D^c R_{kk} - 2D_c(w^c R_{kk}) - 2D_c(K^{cak} R_{ka}) - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + 6w_c K^{ca}_k R_{ak} \\ &\quad - 2K^{ca}_k D_c R_{ka} + 2K^{ca}_k K^i_{ca} R_{ik} + 2K^{ca}_k K^{bk}_c R_{ab} + K^s \nabla_s R_{kk} \\ &= D_c D^c R_{kk} - 2D_c(w^c R_{kk}) - 2D_c(K^{cak}) R_{ka} - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + K^s \nabla_s R_{kk}. \end{aligned} \quad \text{(D.7)}$$

In the last line we again assumed that $\sigma_{(k)} = 0$ and $\theta_{(k)} = 0$. Putting the two terms together leads to some canellations:

$$\begin{aligned} k^i k^j \Box R_{ij} &= 2\nabla_k \nabla_l R_{kk} + 2R_{kmlk} R^m_k + D_c D^c R_{kk} - 2D_c(w^c R_{kk}) \\ &\quad - 2(D_a\theta_{(k)} + R_{kcac})R^a_k + K^s \nabla_s R_{kk}. \end{aligned} \tag{D.8}$$

$\delta X^k_{(4,cov)}$ Simplification The most difficult term in (C.11), which also gives the most interesting results, is

$$k_i D_a D^a X^i_{(2)} = -\frac{1}{2(d-2)}(D_a - w_a)^2 \theta_{(k)} + \frac{1}{2(d-2)} K_{ab} K^{abi} K_i. \tag{D.9}$$

The interesting part here is the first term, so we will take the rest of this section to discuss its variation. The underlying formula is (B.7),

$$\delta w_a = -w^c K_{ac} + R_{klak}. \tag{D.10}$$

From this we can compute the following related variations, assuming that $\theta_{(k)} = 0$ and $\sigma_{(k)} = 0$:

$$\delta(D^a w_a) = D^a R_{klak} + w^a \partial_a \theta_{(k)} - 3D_a(K^{ab}_k w_b) \tag{D.11}$$

$$\delta(w^a D_a \theta_{(k)}) = -3K^{ab}_k w_a D_b \theta_{(k)} + R_{klak} D^a \theta_{(k)} + w^a D_a \dot{\theta}_{(k)} \tag{D.12}$$

$$\delta(D^a D_a \theta_{(k)}) = D^a D_a \dot{\theta} - \partial_a \theta_{(k)} \partial^a \theta_{(k)} - 2P^{jm} R_{kjbm} D^b \theta_{(k)}. \tag{D.13}$$

Here $\dot{\theta}_{(k)} \equiv \delta\theta_{(k)}$ is given by the Raychaudhuri equation. We can combine these equations to get

$$\begin{aligned} \delta\left((D_a - w_a)^2 \theta_{(k)}\right) &= \delta\left(D^a D_a \theta_{(k)}\right) - 2\delta\left(w^a D_a \theta_{(k)}\right) - \delta\left((D_a w^a)\theta_{(k)}\right) + \delta\left(w_a w^a \theta_{(k)}\right) \\ &= -D^a D_a R_{kk} + 2w^a D_a R_{kk} + (D_a w^a) R_{kk} - w_a w^a R_{kk} \\ &\quad - \frac{d}{d-2}(D_a \theta_{(k)})^2 - 2R_{kb} D^b \theta_{(k)} - 2(D\sigma)^2. \end{aligned} \tag{D.14}$$

E The $d = 4$ Case

As mentioned in the main text, many of our calculations are more complicated in even dimensions, though most of the end results are the same. The only nontrivial even dimension we study is $d = 4$, so in this section we record the formulas and special derivations necessary for understanding the $d = 4$ case. Some of these have been mentioned elsewhere already, but we repeat them here so that they are all in the same place.

Log Terms In $d = 4$ we get log terms in the extremal surface, the metric, and the EWN inequality. By looking at the structure of the extremal surface equation, it's easy to see that the log term in in the extremal surface is related to $X^i_{(4)}$ in $d \neq 4$ by first multipling by $4-d$ and then setting $d \to 4$. The result was recorded in (C.13), and we repeat it here:

$$\begin{aligned}
-4X^k_{(4,log)} &= 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) \\
&\quad + K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)} \\
&\quad + k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm} \\
&\quad + 8\kappa\sigma^{ab}_{(k)}C_{kalb}.
\end{aligned} \tag{E.1}$$

There is a similar story for $g^{(4,log)}_{kk}$, which was recorded earlier in (C.7):

$$g^{(4,log)}_{kk} = -\left[\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{24}\nabla^2_k R - \frac{1}{8}k^i k^j \Box R_{ij} - \frac{1}{4}R^{ij}R_{kikj} + \frac{1}{12}RR_{kk}\right]. \tag{E.2}$$

From these two equations, it is easy to see that the log term in $(\delta\bar{X})^2$ has precisely the same form as the subleading EWN inequality (5.1.14) in $d \geq 5$, except we first multiply by $4-d$ and then set $d \to 4$. This results in

$$L^{-2}(\delta\bar{X})^2\big|_{z^2\log z, d=4} = -\frac{1}{4}(D_a\theta_{(k)} + R_{ka})^2 - \frac{1}{4}(D_a\sigma^{(k)}_{bc})^2. \tag{E.3}$$

Note that the Gauss-Bonnet term drops out completely due to special identities of the Weyl tensor valid in $d = 4$ [61]. The overall minus sign is important because $\log z$ should be regarded as negative.

QNEC in Einstein Gravity For simplicity we will only discuss the case of Einstein gravity for the QNEC in $d = 4$, so that the entropy functional is just given by the extremal surface area divided by $4G_N$. At order z^2, the norm of $\delta\bar{X}^\mu$ is formally the same as the expression in other dimensions:

$$L^{-2}(\delta\bar{X})^2\big|_{z^2} = g^{(4)}_{kk} + 2g^{(2)}_{ik}\nabla_k X^i_{(2)} + \nabla_k X^{(2)}_j \nabla_k X^j_{(2)} - \frac{1}{2}X^l_{(2)}\nabla_k R_{kk} + 2\delta(k_i X^i_{(4)cov}). \tag{E.4}$$

Now, though, $X^k_{(4)}$ and $g^{(4)}_{kk}$ are state-dependent and must be related to the entropy and energy-momentum, respectively.

We begin with the entropy. From the calculus of variations, we know that the variation of the extremal surface area is given by

$$\delta A = -\lim_{\epsilon\to 0}\frac{L^3}{\epsilon^3}\int\sqrt{h}\frac{1}{\sqrt{1 + g_{nm}\partial_z\bar{X}^n\partial_z\bar{X}^m}}g_{ij}\partial_z\bar{X}^i\delta X^j. \tag{E.5}$$

A few words about this formula are required. The $\bar{X}^\mu$ factors appearing here must be expanded in ϵ, but the terms without any (n) in their notation do *not* refer to (0), unlike elsewhere in this paper. The reason is that we have to do holographic renormalization carefully at this stage, and that means the boundary conditions are set at $z = \epsilon$. So when we expand out $\bar{X}^\mu$ we will find its coefficients determined by the usual formulas in terms of $X^i_{(0)}$. We need to then solve for $X^i_{(0)}$ in term of $X^i \equiv \bar{X}^i(z=\epsilon)$ re-express the result in terms of X^i alone. Since we are not in a high dimension this task is relatively easy. An intermediate result is

$$\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} = -2\ X^k_{(2)}\big|_{\epsilon^2} - 4\left(X^k_{(4)} - (X_{(2)})^2 X^k_{(2)}\right) - X^k_{(4,log)}. \tag{E.6}$$

The notation on the first term refers to the order ϵ^2 part of $X^i_{(2)}$ that is generated when $X^i_{(2)}$ is written in terms of $\bar{X}^i(z=\epsilon)$. The result of that calculation is

$$\begin{aligned}
-4\ X^k_{(2)}\big|_{\epsilon^2} &= 2X^{(2)}_j K^{jab}K^i_{ab}k_i + k_i D^b D_b X^i_{(2)} + K^m\Gamma^i_{ml}X^l_{(2)}k_i \\
&\quad + g^{ab}_{(2)}K^i_{ab}k_i + P^{kj}R^i_{jmk}X^m_{(2)}k_i + k^m\left(\nabla_j g^{(2)}_{mk} - \frac{1}{2}\nabla_m g^{(2)}_{jk}\right)P^{jk} \\
&= -4X^k_{(4,log)} - 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) - 4g^{(2)}_{km}X^m_{(2)} + K^m\Gamma^i_{ml}X^l_{(2)}k_i.
\end{aligned} \tag{E.7}$$

We have dropped terms of higher order in ϵ. Thus we can write

$$\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} = -3X^k_{(log)} - X^k_{(2)}P^{jm}g^{(2)}_{jm} + 8X^k_{(2)}(X_{(2)})^2 - 2g^{(2)}_{km}X^m_{(2)} - 4X^k_{(4)cov}. \tag{E.8}$$

We will want to take one more variation of this formula so that we can extract $\delta X^k_{(4)cov}$. We can get some help by demanding that the $z^2\log z$ part of EWN be saturated, which states

$$g^{(\log)}_{kk} + 2\delta X^k_{log} = 0. \tag{E.9}$$

Then we have

$$\delta\left(\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0}\right) = \frac{3}{2}g^{(\log)}_{kk} - \delta(X^k_{(2)}P^{jm}g^{(2)}_{jm}) + 8\delta(X^k_{(2)}(X_{(2)})^2) - 2\delta(g^{(2)}_{km}X^m_{(2)}) - 4\delta X^k_{(4)cov}. \tag{E.10}$$

Assuming that $\theta_{(k)} = \sigma_{(k)} = 0$, we can simplify this to

$$\delta\left(\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0}\right) = \frac{3}{2}g^{(\log)}_{kk} - \frac{1}{4}R_{kk}P^{jm}g^{(2)}_{jm} - \frac{1}{4}\nabla_k(\theta_{(l)}R_{kk}) - \frac{1}{2}g^{(2)}_{kl}R_{kk} - 4\delta X^k_{(4)cov}. \tag{E.11}$$

We can combine this with the holographic renormalization formula [73]

$$\begin{aligned}
g^{(4)}_{kk} &= 4\pi G_N L^{-3}T_{kk} + \frac{1}{2}(g^2_{(2)})_{kk} - \frac{1}{4}g^{(2)}_{kk}g^{ij}g^{(2)}_{ij} - \frac{3}{4}g^{(log)}_{kk} \\
&= 4\pi G_N L^{-3}T_{kk} + \frac{1}{8}R^i_k R_{ik} - \frac{1}{16}R_{kk}R - \frac{3}{4}g^{(log)}_{kk}
\end{aligned} \tag{E.12}$$

to get

$$L^{-2}(\delta\bar{X}^i)^2\Big|_{z^2} = 4\pi G_N L^{-3} T_{kk} - \frac{1}{2}\delta\left(\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0}\right). \tag{E.13}$$

After dividing by $4G_N$, we recognize the QNEC.

F Connections to the ANEC

In F we briefly review the connection between the relative entropy and the ANEC. Equation (6.1.2) then implies an interesting connection between the off-diagonal second variation of the entropy and the ANEC. In F we analyze this result in more detail for holographic field theory states dual to perturbative bulk geometries.

ANEC and Relative Entropy

As in Section 6.2, the region $\mathcal{R}$ is a region whose boundary $\partial\mathcal{R}$ lies in the $u = 0$ plane. We also consider a one-parameter family of such regions, indexed by λ, with the convention that increasing λ makes the $\mathcal{R}$ smaller. In this section we will focus on a globally pure state reduced to these regions. The relative entropy (with respect to the vacuum) and its first two derivatives obey the following set of alternating inequalities:

$$S_{\mathrm{rel}} \geq 0, \qquad \frac{dS_{\mathrm{rel}}}{d\lambda} \leq 0, \qquad \frac{d^2 S_{\mathrm{rel}}}{d\lambda^2} \geq 0. \tag{F.1}$$

The first two of these are general properties of relative entropy in quantum mechanics, known as the positivity and monotonicity of relative entropy, respectively. The third inequality is the QNEC.

We can also consider the entropy $\bar{S}$ and relative entropy $\bar{S}_{rel}$ of the complement of $\mathcal{R}$, which we will denote by $\bar{\mathcal{R}}$. Since we specified that the global state is pure, we have $\bar{S} = S$. The set of inequalities obeyed by $\bar{S}_{\mathrm{rel}}$ is

$$\bar{S}_{\mathrm{rel}} \geq 0, \qquad \frac{d\bar{S}_{\mathrm{rel}}}{d\lambda} \geq 0, \qquad \frac{d^2 \bar{S}_{\mathrm{rel}}}{d\lambda^2} \geq 0. \tag{F.2}$$

From (6.2.6) and the analogous equation for $\bar{S}_{\mathrm{rel}}$, together with the monotonicity of relative entropy inequalities, we can conclude

$$\frac{d\bar{S}_{\mathrm{rel}}}{d\lambda} - \frac{dS_{\mathrm{rel}}}{d\lambda} = 2\pi \int d^{d-2}y dv\ T_{vv}\dot{V}(y) \geq 0. \tag{F.3}$$

This is the ANEC, and its connection to relative entropy was first pointed out in [139, 52].

The relation (F.3) has interesting implications. Note that the integral of T_{vv} is completely independent of λ. If we let $\lambda \to \infty$, it must be the case that $dS_{\mathrm{rel}}/d\lambda \to 0$ or else positivity of relative entropy will be violated. Similarly, as $\lambda \to -\infty$ we must have $d\bar{S}_{\rm rel}/d\lambda \to 0$. Then we can say

$$\int_{-\infty}^{\infty} d\lambda \, \frac{d^2 S_{\rm rel}}{d\lambda^2} = \frac{dS_{\rm rel}}{d\lambda}(\infty) - \frac{dS_{\rm rel}}{d\lambda}(-\infty) = 2\pi \int d^{d-2}y dv \; T_{vv} \dot{V}(y). \tag{F.4}$$

From the definition of relative entropy, this means that

$$\int_{-\infty}^{\infty} d\lambda \int d^{d-2}y \; S'' \dot{V}(y)^2 = -\int_{-\infty}^{\infty} d\lambda \int d^{d-2}y d^{d-2}y' \; \frac{\delta^2 S^{\rm od}}{\delta V(y) \delta V(y')} \dot{V}(y) \dot{V}(y'). \tag{F.5}$$

So the diagonal and off-diagonal parts of the second variation entropy contribute equally when integrated over the entire one-parameter family of surface deformations. Since there are two y integrals on the RHS of (F.5), naïvely one might have thought that a limiting case for $\dot{V}(y)$ existed which caused the RHS of this equation to vanish while leaving the LHS finite, but this is not true. We will say more about the order-of-limits involved in the holographic context below. Applying the relation $S''_{vv} = 2\pi \langle T_{vv} \rangle$ we see that, after integration, the off-diagonal variations can be related back to the ANEC:

$$2\pi \int d^{d-2}y dv \; \langle T_{vv} \rangle \dot{V}(y) = -\int_{-\infty}^{\infty} d\lambda \int d^{d-2}y d^{d-2}y' \; \frac{\delta^2 S^{\rm od}}{\delta V(y) \delta V(y')} \dot{V}(y) \dot{V}(y'). \tag{F.6}$$

This is a nontrivial consequence of (6.1.2). Note that $\delta^2 S^{\rm od}/\delta V(y) \delta V(y') \leq 0$ by strong subadditivity [21].

ANEC in a Perturbative Bulk

In this section we will investigate (F.6) in AdS/CFT for perturbative bulk states. Once again, we will drop the contributions of $S_{\rm bulk}$ for simplicity. This amounts to considering coherent states in the bulk.

From (6.3.4), we can see that for perturbative classical bulk states the bulk boost energy completely accounts for the off-diagonal entropy variation. Then from (6.3.7) we get

$$\frac{\delta^2 S^{\rm od}}{\delta V(y_1) \delta V(y_2)} = -2\pi \left(\frac{2^{d-2}\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}} \right)^2 \int \frac{dz d^{d-2}y}{z^{d-1}} \langle T^{\rm bulk}_{vv} \rangle \frac{z^{2d}}{(z^2 + (y-y_1)^2)^{d-1} (z^2 + (y-y_2)^2)^{d-1}} \tag{F.7}$$

As a consequence of (F.6) we then have the equation

$$\int d^{d-2}y dv \; \langle T_{vv} \rangle \dot{V}(y) = \int \frac{dv dz d^{d-2}y}{z^{d-1}} \langle T^{\rm bulk}_{vv} \rangle \dot{\mathcal{V}}(y,z). \tag{F.8}$$

This is a nontrivial matching between the ANEC on the boundary and an associated ANEC in the bulk, made possible by the relationship between $\dot{V}$ and $\dot{\mathcal{V}}$ that comes from solving the extremal surface equation:

$$\dot{\mathcal{V}}(y,z) = \frac{2^{d-2}\Gamma(\frac{d-1}{2})}{\pi^{\frac{d-1}{2}}} \int d^{d-2}y' \frac{z^d}{(z^2 + (y-y')^2)^{d-1}} \dot{V}(y'). \tag{F.9}$$

We can get some intuition for these equations by considering shockwave solutions in the bulk.

Shockwaves Consider a shockwave geometry in the bulk. The bulk stress tensor is [1]

$$\langle T^{\text{bulk}}_{vv}\rangle = E z_0^{d-1}\delta(v)\delta^{d-2}(y)\delta(z-z_0) \tag{F.10}$$

and the boundary stress tensor is

$$\langle T_{vv}\rangle = E\frac{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)z_0^d}{\pi^{\frac{d-1}{2}}(z_0^2+y^2)^{d-1}}\delta(v) \tag{F.11}$$

The parameters z_0 and E characterize the solution. One can see directly that (F.8) holds.

It is also interesting to integrate over a finite range of the deformation parameter. As the range is extended to infinity we recover (F.8), but for finite amounts of deformation we can see how the diagonal and off-diagonal parts of the entropy compete. We take the undeformed surface at $\lambda = 0$ to be the flat plane $V(y) = 0$ and we place the shockwave at $v = v_0$. Then integrating over a range of deformations about zero we find on the boundary

$$\begin{aligned}\int_0^\lambda d\lambda' \int d^{d-2}y\ \langle T_{vv}\rangle \dot{V}(y)^2 &= \int d^{d-2}y\ E\frac{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)z_0^d}{\pi^{\frac{d-1}{2}}(z_0^2+y^2)^{d-1}}\dot{V}(y)\Theta(\lambda\dot{V}(y=0)-v_0)\\ &= E\dot{\tilde{V}}(y=0,z=z_0)\Theta(\lambda\dot{V}(y=0)-v_0).\end{aligned} \tag{F.12}$$

As soon as the integration range crosses $v = v_0$, the total energy jumps from zero to the final answer. On the other hand, in the bulk we get

$$\int_0^\lambda d\lambda' \int \frac{dz d^{d-2}y}{z^{d-1}}\ \langle T^{\text{bulk}}_{vv}\rangle \dot{\tilde{V}}(y,z)^2 = E\dot{\tilde{V}}(y=0,z=z_0)\Theta(\lambda\dot{\tilde{V}}(y=0,z=z_0)-v_0). \tag{F.13}$$

This is a very similar answer, but now the jump does not occur until later: $\dot{\tilde{V}}(y=0,z=z_0)$ will always be less than $\dot{V}(y)$, which means λ has to get larger. How much larger? We can estimate it by looking at the example of a bump function deformation with $\dot{V}(y) = 1$ over a region of area $\mathcal{A} \ll z_0^{d-2}$ and zero elsewhere. Then the boundary energy will register at $\lambda = v_0$, while the bulk energy will register at

$$\lambda = \frac{\pi^{\frac{d-1}{2}}}{2^{d-2}\Gamma(\frac{d-1}{2})}\frac{z_0^{d-2}}{\mathcal{A}}v_0 \gg v_0\ . \tag{F.14}$$

So for very narrow deformations, the off-diagonal contributions to the entropy can only be seen when integrated over a large range of the deformation parameter. From the boundary point of view, the parameter z_0 controls how diffuse the energy is in the y-directions. It is a measure of the nonlocality of the state. The off-diagonal entropy variations are sensitive to this nonlocality.

Note that the order of limits we have discovered here is worth repeating. If we take $\mathcal{A} \to 0$ before taking $\lambda \to \infty$ then our integration will only be sensitive to the diagonal entropy variation (i.e., the boundary stress tensor) and we will find apparent violations of (F.6). The reason is that there are important contributions to the off-diagonal entropy variations when $\lambda \sim z_0^{d-2}/\mathcal{A}$, where z_0 controls the level of nonlocality in the state.

Superpositions of Shockwaves At linear order in the bulk perturbations we can take superpositions of shockwaves. This allows us to create any bulk and boundary bulk stress tensor profile along the $u = 0$ plane, and in that sense represents the most general state for the purpose of this calculation. The bulk and boundary stress tensors would be

$$T_{vv}^{\text{bulk}}(y, z, v) = z^{d-1}\rho(y, z, v) \tag{F.15}$$

and

$$T_{vv}(y, v) = \frac{2^{d-2}\Gamma\left(\frac{d-1}{2}\right)}{\pi^{\frac{d-1}{2}}} \int d^{d-2}y' dz' \rho(y', z', v) \frac{(z')^d}{((z')^2 + (y-y')^2)^{d-1}} \tag{F.16}$$

The single shockwave is the special case $\rho = E\delta(v)\delta^{d-2}(y)\delta(z - z_0)$. We can repeat some of the calculations we did before, but qualitatively the results will be the same. The deformed bulk extremal surface always “lags behind” the deformed entangling surface in a way that depends on z and the width of the deformation, and as a result the bulk energy flux at finite deformation parameters will always be less than the boundary energy flux. Taking the deformation width to zero at finite deformation parameters will cause the bulk energy flux to drop to zero. It would be interesting to characterize this behavior directly in the field theory without the bulk picture.

G Free and Weakly-Interacting Theories

Our conjectures (6.1.7) and (6.1.2) are only meant to apply to interacting theories. In this appendix we will explain how the null-null relation (6.1.2) is violated in free theories, and indicate how it might be fixed when interactions are included.

The Case of Free Fields

The case of free fields for entangling surfaces restricted to $u = 0$ was analyzed extensively in [27], and we will make use of that analysis here. As in Section 6.2 we have a one-paramter family of regions indexed by λ. The deformation velocity $\dot{V}(y)$ is taken to be a unit step-function with support on a small region of area $\mathcal{A}$ in the y-directions. The crucial point is to focus attention on the pencil of the $u = 0$ plane that is the support of $\dot{V}(y)$. As λ varies, the entangling surface moves within this pencil but stays fixed outside of it.

The State and the Entropy For the purpose of constructing the state, we can model the full theory as a $1+1$-dimensional massless chiral boson living on the pencil, together with an auxiliary system consisting of the rest of the $u=0$ plane. This is the formalism of null quantization, which is reviewed in [27].

There are two facts we're going to use to write down the sate $\rho(\lambda)$ on the pencil+auxiliary system. First, in the limit of small $\mathcal{A}$, the state on the pencil becomes approximately disentangled from the auxiliary system. The fully-disentangled part $\mathcal{A}^0$ part of the state looks like the vacuum, while the leading correction goes like $\mathcal{A}^{1/2}$ and consists of single-particle states on the pencil entangled with states of the auxiliary system. The second fact is that we can always translate our state in the pencil by an amount λ so that the entangling surface is at the origin and the operators which create the state are displaced by an amount λ. From their original positions. A coordinate system where the entangling surface is fixed is preferable. Putting these facts together lets us write

$$\rho(\lambda) = \rho_{\text{vac}} \otimes \left(\sum_i e^{-2\pi K_i} |i\rangle\langle i| \right) + \mathcal{A}^{1/2} \sum_{i,j} \rho_{ij}^{(1/2)}(\lambda) \otimes \left(e^{-\pi(K_i+K_j)/2} |i\rangle\langle j| \right) + \cdots \tag{G.1}$$

The states $|i\rangle$ of the auxiliary system are merely those which diagonalize the $\mathcal{A}^0$ part of ρ, and the K_i are numbers specifying the eigenvalues.

As indicated above the state $\rho_{ij}^{(1/2)}(\lambda)$ should be interpreted as a state on the half-line $x>0$. We can write this state in terms of a Euclidean path integral in the complex plane:

$$\rho_{ij}^{(1/2)}[\phi^-, \phi^+] = \int_{\phi(x^+)=\phi^+}^{\phi(x^-)=\phi^-} \mathcal{D}\phi \; \mathcal{O}_{ij}(\lambda) e^{-S_{\text{E}}}, \tag{G.2}$$

where $\phi(x^{\pm})$ refers to boundary conditions just above/below the positive real axis. The insertion $\mathcal{O}_{ij}(\lambda)$ is a single-field insertion which specifies the state:

$$\mathcal{O}_{ij}(\lambda) = \int dz d\bar{z} \; \psi_{ij}(z, \bar{z}) \partial\phi(z-\lambda). \tag{G.3}$$

As in [27] we will normalize our field so that $\langle \partial\phi(z)\partial\phi(0)\rangle_{\text{vac}} = -1/z^2$ and $T_{vv} = (\partial\phi)^2/4\pi\mathcal{A}$. Then one can show that $Q \equiv S''_{vv} - 2\pi T_{vv}$ is given by

$$Q(\lambda) = -\frac{1}{2}\sum_{ij} \left| \int dx d\tau \; (z-\lambda)^{-2+i\alpha_{ij}} \psi_{ij}(x,\tau) \right|^2 \frac{\pi(1+\alpha_{ij}^2)\alpha_{ij}}{\sinh \pi\alpha} e^{2\pi\alpha_{ij}} \tag{G.4}$$

where if $z = re^{i\theta}$ with $0 \le \theta < 2\pi$ then

$$z^{i\alpha} = r^{i\alpha} e^{-\alpha\theta}. \tag{G.5}$$

The quantity Q is manifestly negative, as required by the QNEC, but it is not zero.

Recovering the ANEC In Appendix F we showed how one can recover the ANEC by integrating the QNEC on a globally pure state. In the present context, we don't have any off-diagonal contributions to the entropy. Instead we have the function Q, and repeating the argument above would lead us to conclude

$$\int_{-\infty}^{\infty} d\lambda\ Q(\lambda) = -2\pi \int d\lambda\ T_{vv}(\lambda). \tag{G.6}$$

We can check this equation by integrating (G.4). Note that the assumption of global purity that was used in Appendix F is crucial: the expectation value of $T_{vv}(\lambda)$ depends only on the part of the state proportional to $\mathcal{A}$, which we have not specified and in principle has many independent parameters. For a globally pure state there is a relationship between that part of the state and the $\mathcal{A}^{1/2}$ part of the state which we must exploit.

In the pencil+auxiliary model, the global Hilbert space consists of the full pencil plus a doubled auxiliary system. The doubling allows the auxiliary state to be purified. Let the global pure state by $|\Psi\rangle$. Then we have

$$|\Psi\rangle = |\mathrm{v}ac\rangle \otimes \left(\sum_i e^{-\pi K_i} |i\rangle \otimes |i\rangle\right) + \mathcal{A}^{1/2} \sum_{i,j} e^{-\pi\alpha_{ij}/2} |\Psi_{ij}\rangle \otimes |i\rangle \otimes |j\rangle + \cdots \tag{G.7}$$

Any subsequent terms will not affect the ANEC. The factor of $\exp(-\pi\alpha_{ij}/2)$ is purely for future convenience, and the $|\Psi_{ij}\rangle$ are not necessarily normalized. The expectation value of the ANEC operator in this state is given by

$$2\pi \int d\lambda\ \langle T_{vv}(\lambda)\rangle_\Psi = 2\pi\mathcal{A} \sum_{i,j} e^{-\pi\alpha_{ij}} \int d\lambda\, \langle \Psi_{ij}|\, T_{vv}(\lambda)\, |\Psi_{ij}\rangle\,. \tag{G.8}$$

We can make contact with our earlier formulas by computing the density matrix $|\Psi\rangle\langle\Psi|$ and tracing over the second copy of the auxiliary system. We find that

$$\rho_{ij}^{(1/2)} = \mathrm{Tr}_{x<0}\left(|\Psi_{ij}\rangle\langle \mathrm{v}ac| + |\mathrm{v}ac\rangle\langle\Psi_{ji}|\right). \tag{G.9}$$

This lets us identify the part of $\mathcal{O}_{ij}$ in the lower half-plane as the operator which creates $|\Psi_{ij}\rangle$. Then, in our previous notation, we find

$$2\pi \int d\lambda\ \langle T_{vv}(\lambda)\rangle_\Psi = 4\pi i \sum_{i,j} e^{-\pi\alpha_{ij}} \int dx d\tau dx' d\tau' \frac{\psi_{ij}(x,\tau)\psi_{ij}(x',\tau')^*}{(z-w^*)^3}\Theta(-\tau)\Theta(-\tau'). \tag{G.10}$$

Our job now is to reproduce this by integrating (G.4) with respect to λ. The main identity we will need is

$$\int_{-\infty}^{\infty} \frac{d\lambda}{(z-\lambda)^{2-i\alpha_{ij}}(w^*-\lambda)^{2+i\alpha_{ij}}} = \frac{4ie^{-2\pi\alpha_{ij}}\sinh\pi\alpha_{ij}}{\alpha_{ij}(1+\alpha_{ij}^2)(w^*-z)^3}\left(e^{\pi\alpha_{ij}}\Theta(\tau)\Theta(\tau') - e^{-\pi\alpha_{ij}}\Theta(-\tau)\Theta(-\tau')\right). \tag{G.11}$$

Using this formula, the integral of (G.4) splits into two terms. We may combine them by exchanging i and j in the first term, leaving us with

$$\int d\lambda\, Q(\lambda) = -2\pi i \sum_{ij} \int dx d\tau dx' d\tau' \, \frac{\psi_{ij}(x,\tau)\psi_{ij}(x',\tau')^*}{(w^* - z)^3} \left(e^{\pi\alpha_{ij}}\Theta(\tau)\Theta(\tau') - e^{-\pi\alpha_{ij}}\Theta(-\tau)\Theta(-\tau')\right)$$
$$= -4\pi i \sum_{ij} e^{-\pi\alpha_{ij}} \int dx d\tau dx' d\tau' \, \frac{\psi_{ij}(x,\tau)\psi_{ij}(x',\tau')^*}{(z - w^*)^3} \Theta(-\tau)\Theta(-\tau') \tag{G.12}$$

Coherent States For coherent states we obtain a correspondence between Q and T_{vv} without integrating over λ. This must be true because coherent states satisfy $S''_{vv} = 0$, but it is reassuring to see it happen explicitly. In a coherent state of the original d-dimensional theory, the pencil and auxiliary system factorize and the pencil is in a $1+1$-dimensional coherent state. In other words, we have

$$\rho(\lambda)[\phi^-, \phi^+] = \left(\int_{\phi(x^+)=\phi^+}^{\phi(x^-)=\phi^-} \mathcal{D}\phi \; e^{-S_{\mathrm{E}} + \mathcal{A}^{1/2}\mathcal{O}(\lambda)}\right) \otimes \left(\sum_i e^{-2\pi K_i} |i\rangle\langle i|\right). \tag{G.13}$$

We can obtain Q for this state by taking the general equation (G.4) specializing to the case where $\psi_{ij} = \psi\delta_{ij}\exp(-\pi K_i)$. Making use of the normalization condition $\sum_i \exp(-2\pi K_i) = 1$ we find the simple expression

$$Q_{coherent}(\lambda) = -\frac{1}{2}\left|\int dx d\tau \; \frac{\psi(x,\tau)}{(z-\lambda)^2}\right|^2 = -\frac{1}{2\mathcal{A}}\langle\partial\phi(\lambda)\rangle^2_{coherent}. \tag{G.14}$$

We recognize this as simply $-2\pi\langle T_{vv}\rangle_{coherent}$, as expected.

Weakly Interacting Theories and Effective Field Theories

In the main text we provided evidence for that $S''_{vv} = 2\pi\langle T_{vv}\rangle$ for interacting theories, but in the previous section we explained that for free theories $Q = S''_{vv} - 2\pi\langle T_{vv}\rangle$ was nonzero, and in fact could be quite large. In this section we will show how we can transition from $S''_{vv} \neq 2\pi\langle T_{vv}\rangle$ to $S''_{vv} = 2\pi\langle T_{vv}\rangle$ when a weak coupling is turned on.[2]

The essential point is that one should always consider the total variation $d^2S/d\lambda^2$ as the primary physical quantity. S''_{vv} is a derived quantity obtained by considering a limiting case of arbitrarily thin deformations. However, a weakly-coupled effective field theory in the IR comes with a cutoff scale ϵ, and we cannot reliably compute $d^2S/d\lambda^2$ for deformations of width $\ell \lesssim \epsilon$. Now we will see how this can resolve the issue.

In the free theory, as we have explained above, the second functional derivative of the entropy has the form

$$\frac{\delta^2 S_{\mathrm{free}}}{\delta V(y)\delta V(y')} = 2\pi\langle T_{vv}\rangle\delta^{(d-2)}(y-y') + Q\delta^{(d-2)}(y-y') + \frac{\delta^2 S^{\mathrm{od}}}{\delta V(y)\delta V(y')}. \tag{G.15}$$

[2]We thank Thomas Faulkner for first pointing out the arguments we present in this section.

The function Q is related to the square of the expectation value of the field $\partial\phi$. This is especially obvious in the formula for the coherent state, (G.14), but the more general formula is essentially of the same form. In a free theory $(\partial\phi)^2$ has dimension d and is exactly of the right form to contribute to a δ-function. This fact was touched upon in the Introduction. When we turn on a weak coupling g, the dimension of ϕ will shift to $\Delta_\phi = (d-2)/2+\gamma(g)$.[3] There will still be a term in the second variation of the entropy associated to $(\partial\phi)^2$, which we will call Q_g, but now it no longer comes with a δ-function:

$$\frac{\delta^2 S_g}{\delta V(y)\delta V(y')} = 2\pi T_{vv}\delta^{(d-2)}(y-y') + Q_g f_g(y-y') + \frac{\delta^2 S^{\rm od}}{\delta V(y)\delta V(y')}. \tag{G.16}$$

Here f_g is some function of mass dimension $d-2-2\gamma$ which limits to a δ-function as $g\to 0$, such that $f_g(y)\sim \gamma/y^{d-2-2\gamma}$. So the Q_g term has migrated from the δ-function to the off-diagonal part of the entropy variation.

Now consider integrating (G.16) twice against a deformation profile of width ℓ and unit height to get a total second derivative of the entropy. Suppose that ℓ is very small compared to the length scales of the state, but still large compared to the cutoff ϵ. Then we have

$$\frac{d^2 S_g}{d\lambda^2} = 2\pi T_{vv}\ell^{d-2} + Q_g \ell^{d-2+2\gamma} + \frac{d^2 S^{\rm od}}{d\lambda^2}. \tag{G.17}$$

We can write $Q_g \sim QM^{2\gamma}$, where M is a mass scale characterizing the state and Q is what we get in the $g\to 0$ limit. So at weak coupling, we can say that

$$Q_g \ell^{d-2+2\gamma} \sim Q\ell^{d-2}\left(1+2\gamma\log M\ell + \cdots\right). \tag{G.18}$$

Thus we find that the answer for the weakly-coupled theory is approximately the same as for the free theory, as long as $\gamma \log M\ell \ll 1$. The smallest we can make ℓ is of order the cutoff ϵ, and the condition that $\gamma\log M\epsilon$ remain small is analogous to the problem of large logarithms in perturbation theory. The renormalization group is typically used to get around the problem of large logarithms, and it would be interesting to apply those same ideas to the present situation.

This argument hints that for general effective field theories S''_{vv} may not have a good operational meaning in terms of physical observables. The relevant condition for isolating the δ-function is that $(M\ell)^{2\gamma}\ll 1$ should be possible within the effective description. Clearly this can be done in an exact CFT with finite anomalous dimensions, but it should also be possible if the theory is approximately given by an interacting CFT over some large range of length scales. For instance, if an interacting CFT is weakly coupled to gravity and we consider states with energy M much less than the Planck scale then it should be possible to have $(M\ell)^{2\gamma}\ll 1$ while maintaining $\ell \gg \ell_{Planck}$.

Finally, a more precise version of the arguments given above can be given by interpreting the second functional derivative of the entropy as an OPE. We hope to use these techniques to find the exact form of f_g in future work [8].

[3] We treat g and γ as fixed numbers that do not themselves depend on scale. A more complete treatment that incorporates the RG flow of the coupling would be interesting.

H Modified Ward identity

In this Appendix, we prove the following identity:

$$\int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') \hat{D}_+(y) T_{--}(w,\bar{w},0)\rangle = -\partial_{\bar{w}} \langle \Sigma_n^0 \hat{D}_+(y) T_{--}(w,\bar{w},0)\rangle. \tag{H.1}$$

This is similar to the defect CFT ward identity of [15] except there is another insertion of the displacement operator. A priori it is not obvious that some form of the Ward identity carries through in the case where more than one operator is a defect operator. We will argue essentially that the second insertion of a $\hat{D}_+$ just comes along for the ride.

To show this, first we write the displacement operator as a stress tensor integrated around the defect:

$$\hat{D}_+(y) = i \oint d\bar{z}\ T_{++}(0,\bar{z},y) \tag{H.2}$$

where we have suppressed the sum over replicas to avoid clutter. We will then argue that the following equality holds

$$\begin{aligned} &i \lim_{\varepsilon\to 0} \oint_{\varepsilon>|\bar{z}|} d\bar{z} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') T_{++}(0,\bar{z},y) T_{--}(w,\bar{w},0)\rangle \\ &= \int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') \hat{D}_+(y) T_{--}(w,\bar{w},0)\rangle \end{aligned} \tag{H.3}$$

for some appropriate $\varepsilon > 0$ that acts as the cutoff $|y' - y| > \varepsilon$.

To see this, simply note that we can replace $T_{++}(0,\bar{z},y)$ by a sum over local defect operators at y using the bulk-defect OPE. The important point is that this OPE converges because the $\bar{z}$ contour is always inside of the sphere of size ε (by construction). We can take $|\bar{z}|$ to be arbitrarily small by making the size of the $\bar{z}$ contour as small as we like. The $\bar{z}$ integral outside now simply projects the sum onto the displacement operator since we only consider the leading twist $d-2$ operators in the lightcone limit. Explicitly, we will be left with

$$\begin{aligned} &i \lim_{\varepsilon\to 0} \oint_{\varepsilon>|\bar{z}|} d\bar{z} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') T_{++}(0,\bar{z},y) T_{--}(w,\bar{w},0)\rangle \\ &= \lim_{\epsilon\to 0} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') \hat{D}_+(y) T_{--}(w,\bar{w},0)\rangle. \end{aligned} \tag{H.4}$$

Note that perturbatively around $n = 1$, the integral over $|y - y'| > \epsilon$ will miss the delta function contribution to the $\hat{D}_+ \times \hat{D}_+$ OPE. Non-perturbatively away from $n = 1$, however, there are no delta-function singularities in $|y - y'|$ present in the $\hat{D}_+ \times \hat{D}_+$ OPE. In what follows, we must be careful to take $\epsilon \to 0$ *before* taking $n \to 1$.

Using this identity, we can view the displacement-displacement-bulk three point function as the contour integral of a displacement-bulk-bulk three point function. We can then use the regular displacement operator Ward identity on the latter three point function. This Ward identity follows from general diffeomorphism invariance [15]. To do this, define the deformation vector field

$$\xi(y') = f(y')\partial_+ \text{ with } f(y') = \Theta(|y'-y|-\varepsilon). \tag{H.5}$$

For this deformation, the Ward identity takes the form

$$\begin{aligned} &i\oint_{\varepsilon>|\bar z|} d\bar z \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat D_+(y') T_{++}(0,\bar z,y) T_{--}(w,\bar w,0)\rangle \\ &= -f(0)\partial_{\bar w}\langle \Sigma_n^0 \hat D_+(y) T_{--}(w,\bar w,0)\rangle - i\oint d\bar z f(y)\partial_{\bar z}\langle \Sigma_n^0 T_{++}(0,\bar z,y) T_{--}(w,\bar w,0)\rangle \\ &- i\int_{\mathcal{M}_n} d^d x' \oint d\bar z\ \langle T_{++}(0,\bar z,y) T_{--}(w,\bar w,0) T^{\mu\nu}(x')\partial_\mu \xi_\nu(x')\rangle \end{aligned} \tag{H.6}$$

where $\mathcal{M}_n$ is the full replica manifold.

The second term on the right hand side of the equality vanishes because $f(y)=0$. Since $f(0)=1$ by construction we just need to argue that the last term in (H.6) vanishes.

Arguing the last term vanishes

It is tempting at this stage to integrate by parts on the last term and conclude that this vanishes as one sends $\varepsilon\to 0$. Unfortunately, the last term in (H.6) can produce $1/\varepsilon$ enhancements due to T_{i+} operator coming ε close to T_{++}. Therefore one must take care to first do the x' integral and then take the $\varepsilon\to 0$ limit when evaluating this term.

To do so, note that

$$T^{\mu\nu}(x')\partial_\mu\xi_\nu(x') = \frac{1}{2}T_{i+}(x')\hat n^i \delta(|y'-y|-\varepsilon) \tag{H.7}$$

where $\hat n^i = (y'-y)^i/|y'-y|$. We then have the following

$$\begin{aligned} &\int_{\mathcal{M}_n} d^d x' \oint d\bar z\ \langle T_{++}(0,\bar z,y) T_{--}(w,\bar w,0) T^{\mu\nu}(x')\partial_\mu\xi_\nu(x')\rangle \\ &= \frac{1}{2}\varepsilon^{d-3}\int \rho' d\rho' d\theta' \oint d\bar z \int d^{d-3}\vartheta'\ \hat n^i \langle T_{++}(0,\bar z,y) T_{--}(w,\bar w,0) T_{i+}(|\vec y+\vec\varepsilon|, \vartheta'_{\vec\varepsilon}, \rho' e^{-i\theta'}, \rho' e^{-i\theta'})\rangle \end{aligned} \tag{H.8}$$

where $|\vec\varepsilon| = \varepsilon$. In going to the second line we have done the coordinate transformation $x'^+ = \rho' e^{-i\theta'}$, $x'^- = \rho' e^{i\theta'}$ because we are in the Euclidean section, and in going to the last line we have written y' in spherical coordinates on the defect. At this point we can safely send $w,\bar w\to 0$ so that T_{--} is simply fixed at the origin. Then, in particular, let us focus on

$$\int d\theta' \oint d\bar z\ \langle T_{++}(0,\bar z,y) T_{--}(0) T_{i+}(|\vec y+\vec\varepsilon|, \vartheta'_{\vec\varepsilon}, \rho' e^{-i\theta'}, \rho' e^{-i\theta'})\rangle. \tag{H.9}$$

It is easy to see that this identically vanishes from the boost weights of the quantities involved. Specifically, T_{++} will yield a factor of $e^{2i\theta'}$, T_{i+} will yield a factor of $e^{i\theta'}$, T_{--} does not have a boost weight since it is fixed at the origin, and the measure $d\bar{z}$ will yield a factor of $e^{-i\theta'}$ so overall we will have $\int_0^{2\pi} d\theta' e^{i\theta'} = 0$. Therefore (H.8) is zero for any ε.

Thus, the identity in (H.6) becomes

$$\begin{aligned} &i \lim_{\epsilon \to 0} \oint_{\varepsilon > |\bar{z}|} d\bar{z} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') T_{++}(0,\bar{z},y) T_{--}(w,\bar{w},0)\rangle \\ &= -\partial_{\bar{w}} \langle \Sigma_n^0 \hat{D}_+(y) T_{--}(w,\bar{w},0)\rangle \end{aligned} \tag{H.10}$$

which, using (H.3), proves (H.1).

I Analytic Continuation of a Replica Three Point Function

In this section, we analytically continue a general $\mathbb{Z}_n$-symmetrized three point function of the form[4]

$$\mathcal{A}_n^{(3)} = n \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \mathrm{Tr}\left[e^{-2\pi n H} \mathcal{T} \mathcal{O}_a(0) \mathcal{O}_b(\tau_{ba} + 2\pi j) \mathcal{O}_c(\tau_{ca} + 2\pi k)\right] \tag{I.1}$$

where H is the vacuum modular Hamiltonian for the Rindler wedge and $\mathcal{T}$ denotes Euclidean time ordering with respect to this Hamiltonian.

Following [48], we begin by rewriting the the j-sum as as a contour integral

$$\frac{n}{2\pi i} \sum_{k=0}^{n-1} \oint_{C_b} ds_b \frac{\mathrm{Tr}\left[e^{-2\pi n H} \mathcal{T} \mathcal{O}_a(0) \mathcal{O}_b(-is_b) \mathcal{O}_c(2\pi k + \tau_{ca})\right]}{(e^{s_b - i\tau_{ba}} - 1)} \tag{I.2}$$

where the contour C_b wraps the n poles at $s_b = i(2\pi j + \tau_{ba})$ for $j = 0, ..., n-1$. We will now unwrap the s_b contour integral in the complex plane, but will need to be careful as the analytic structure of the integrand in (I.2) is non-trivial as a function of s_b; the integrand has poles at $s_b = i(2\pi j + \tau_{ba})$ and light-cone branch cuts lying along the lines Im $s_b = 0, 2\pi n$ and Im $s_b = 2\pi k + \tau_{ca}$ for a fixed k. The first two branch cuts were discussed in [48]. The third (middle in the figure) branch cut arises from singularities due to $\mathcal{O}_b$ and $\mathcal{O}_c$ lying on the same light-cone.

[4]Note that we are writing this as a thermal three point function on $\mathbb{H}_{d-1} \times S_1$, which is related to the flat space replica answer via conformal transformation. For a review of the relevant conformal factors, which we suppress for convenience, see [48].

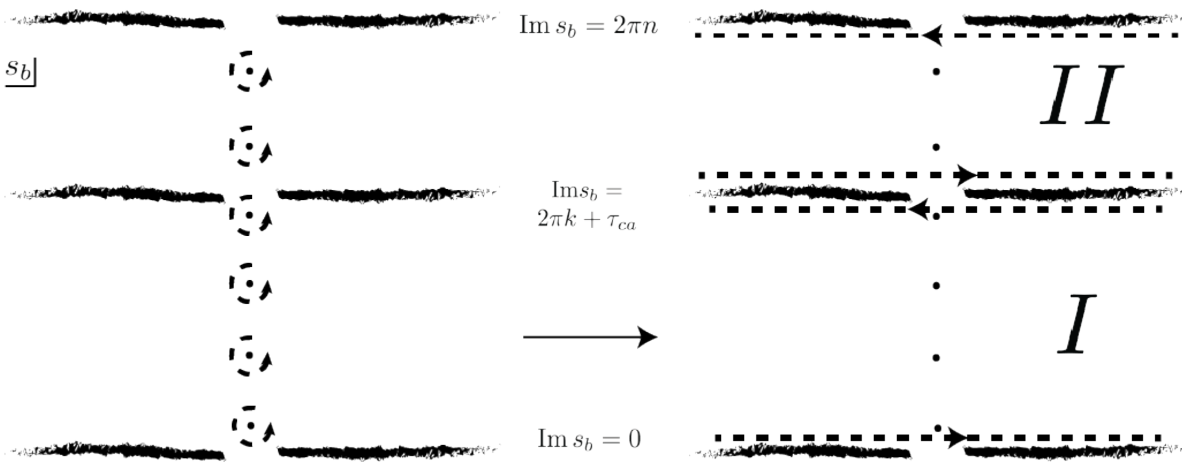

Figure 8.1: The analytic structure of the integral in equation (I.2) represented in the s_b plane for fixed $s_k = i(2\pi k + \tau_{ca})$ for $n = 6$. The dots represent poles at $s_b = i(2\pi j + \tau_{ba})$ and the fuzzy lines denote light-cone branch cuts. The bottom and top branch cuts (which are identified by the KMS condition) arise from $\mathcal{O}_b$ becoming null separated from $\mathcal{O}_a$ and the middle branch cut arises from $\mathcal{O}_b$ becoming null separated from $\mathcal{O}_c$. Note that in this figure, $k = 3$ and $\tau_{ca} > \tau_{ba} > 0$. We start with the contour C_b represented by the dashed lines encircling the poles at $s_b = i(2\pi j + \tau_{ba})$ and unwrap so that it just picks up contributions from the branch-cuts. Region I corresponds to the ordering $\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c$ whereas region II corresponds to $\mathcal{O}_a\mathcal{O}_c\mathcal{O}_b$.

.

We can unwrap the C_b contour now so that it hugs the branch cuts as in the right-hand panel of Figure 8.1. We will then be left with a sum of four Lorentzian integrals

$$\begin{aligned}&\frac{n}{2\pi i}\sum_{k=0}^{n-1}\mathrm{Tr}\Bigg[e^{-2\pi n H}\int_{-\infty}^{\infty}ds_b\times\\&\frac{\mathcal{O}_a(0)\mathcal{O}_b(-is_b+\epsilon_j)\mathcal{O}_c(2\pi k+\tau_{ca})}{(e^{s_b-i\tau_{ba}}-1)}-\frac{\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi ik+\tau_{ca}-\epsilon)\mathcal{O}_c(2\pi k+\tau_{ca})}{(e^{s_b+2\pi ik+\tau_{ca}-i\epsilon-i\tau_{ba}}-1)}\\&+\frac{\mathcal{O}_a(0)\mathcal{O}_c(2\pi k+\tau_{ca})\mathcal{O}_b(-is_b+2\pi k+\tau_{ca}+\epsilon)}{(e^{s_b+2\pi ik+\tau_{ca}+i\epsilon-i\tau_{ba}}-1)}-\frac{\mathcal{O}_a(0)\mathcal{O}_c(2\pi k+\tau_{ca})\mathcal{O}_b(-is_b+2\pi n-\epsilon)}{(e^{s_b+i2\pi n-i\epsilon-i\tau_{ba}}-1)}\Bigg],\end{aligned} \tag{I.3}$$

where we have set $2\pi k+\tau_{ca} = -is_c$ since the C_c contour still wraps the poles at these values.

We now need to make a choice about how to do the analytic continuation in n. The usual prescription, which was advocated for in [48], is to set $e^{2\pi in} = 1$ in the last term of (I.3). We will follow this but also make one other choice. In the second and third terms in the integrand of (I.3) we make the choice to set $e^{2\pi ik} = 1$ for all $k = 0, ..., n-1$.

Making this analytic continuation, we can now re-write the k-sum as a contour integral over s_c along some contour C_c. Unwrapping this s_c contour into the Lorentzian section, and after repeated use of the KMS condition to push operators back around the trace, we land on the relatively simple formula

$$\mathcal{A}_n^{(3)} = \frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \,\mathrm{Tr}\left[e^{-2\pi n H}\left(\frac{[[\mathcal{O}_a(0),\mathcal{O}_b(-is_b)],\mathcal{O}_c(-is_c)]}{(e^{s_b-i\tau_{ba}}-1)(e^{s_c-i\tau_{ca}}-1)} - \frac{[\mathcal{O}_a(0),[\mathcal{O}_b(-is_b-is_c),\mathcal{O}_c(-is_c)]]}{(e^{s_b+i\tau_{ca}-i\tau_{ba}}-1)(e^{s_c-i\tau_{ca}}-1)}\right)\right] \tag{I.4}$$

In deriving this formula, we have assumed $\tau_{ba} > 0$ and $\tau_{ca} > 0$ but we have not yet assumed any relationship between τ_{ba} and τ_{ca}. This formula is the full answer. One could stop here, but we will massage this formula into a slightly different form for future convenience. Instead of following [48] and applying ∂_n at this stage, which drops down powers of H, we will use a slightly different (although equivalent) technique.

We first focus on re-writing the two Lorentzian integrals in region I of Figure 8.1 as one double integral.

Region I

Before re-writing the k-sum as a contour integral, the integrals in region I are[5]

$$\frac{n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b\left(\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b-i\tau_{ba}}-1)} - \frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi k+\tau_{ca}-\epsilon)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b+i\tau_{ca}-i\tau_{ba}}-1)}\right) \tag{I.5}$$

where as before we have set $e^{2\pi ik} = 1$ in the second term. The goal will be to make the denominators in these two terms the same so that we may combine their numerators. We will try to shift the s_b contour in the second term by an amount $-i\tau_{ca}$, making sure not to cross any poles or branch cuts. To make our lives easier, we will assume a fixed ordering of the operators. For now, we will pick $\tau_{ca} > \tau_{ba} > 0$. Note that any other ordering can be reached just by exchanging the a, b, c labels.

In this ordering, sending $s_b \to s_b - i\tau_{ca}$ crosses a pole at Im $s_b = 2\pi k + \tau_{ba}$. This contour shift is illustrated in Figure 8.2. After doing this shift, we get

$$\frac{n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b\left(\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n - \langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi k)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b-i\tau_{ba}}-1)}\right)$$
$$+\,\theta(\tau_{cb}) \times (\text{terms with } j = k). \tag{I.6}$$

where we will mostly neglect the extra term coming from picking up the pole since it will not be important for most calculations we are interested in. We will refer to these terms as the "replica diagonal terms" since they arise from terms in the double sum over j, k in (I.1) where $j = k$.

[5]For ease of notation, we have switched to $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_n = \mathrm{Tr}[e^{-2\pi n H}\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$.

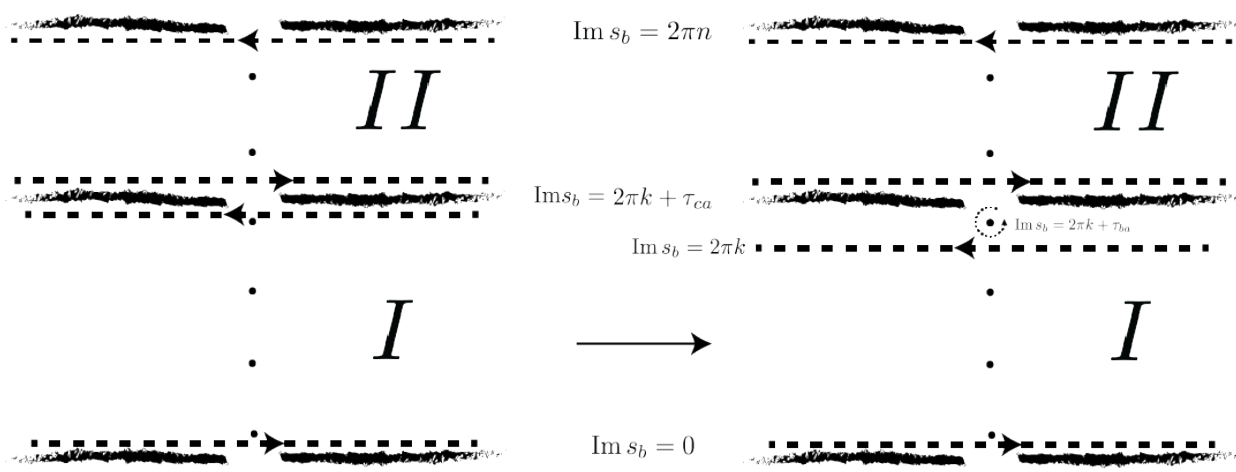

Figure 8.2: This figure illustrates the contour shift $s_b \to s_b - i\tau_{ca}$ done at the cost of picking up the pole at $s = i(2\pi k + \tau_{ba})$ when $\tau_{cb} = \tau_{ca} - \tau_{ba} > 0$.

The numerator for the first term in equation (I.6) then looks like the integral of a total derivative in some auxiliary parameter t_b which we write as

$$\frac{-n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b \int_0^{i2\pi k} dt_b \left(\frac{\frac{d}{dt_b}\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(2\pi k + \tau_{ca})\rangle_n}{(e^{s_b - i\tau_{ba}} - 1)}\right). \tag{I.7}$$

Since t_b shows up on equal footing with s_b in the numerator, we see we can re-write the derivative in t_b as one in s_b. Integrating by parts and dropping the boundary terms[6], we get

$$\frac{-n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b \int_0^{i2\pi k} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(2\pi k + \tau_{ca})\rangle_n}{4\sinh^2((s_b - i\tau_{ba})/2)}. \tag{I.8}$$

We are now ready, as above, to turn the sum over k into a contour integral over some Lorentzian parameter s_c. We can then execute the same trick as before: we re-write two terms as the boundary terms of one integral in some new auxiliary parameter t_c. After all of this, the answer we find is the relatively simple result for region I

$$\begin{aligned}\text{region I} =& \frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_0^{s_c + t_c} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\ &+ \theta(\tau_{cb}) \times (\text{terms with } j = k).\end{aligned} \tag{I.9}$$

Note that the quadruple integral term is manifestly order $n-1$ because of the limits on the t_c integral.

[6]We will drop boundary terms at large Lorentzian time everywhere throughout this discussion, as we expect thermal correlators to fall off sufficiently quickly [48].

Region II

In region II of Figure 8.1, the calculations are exactly analogous, except now the ordering of the operators is different. We find that (up to terms that again come from picking up specific poles) the answer for region II is

$$\begin{aligned}
&\text{region II} = \\
&\frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_{s_c+t_c+i2\pi}^{i2\pi n} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\mathcal{O}_b(-is_b - it_b)\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\
&+ \theta(\tau_{bc}) \times (\text{terms with } j = k).
\end{aligned} \tag{I.10}$$

Combining Regions I and II

Adding the Region I and Region II contributions, we get for the non-replica diagonal contributions to $\mathcal{A}_n^{(3)}$

$$\begin{aligned}
&\frac{n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_0^{s_c+t_c} dt_b \frac{\langle [\mathcal{O}_b(-is_b - it_b), \mathcal{O}_a(0)]\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\
&+\frac{n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_{s_c+t_c}^{s_c+t_c+i2\pi(1-n)} dt_b \frac{\langle \mathcal{O}_b(-is_b - it_b)\mathcal{O}_a(0)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)}
\end{aligned} \tag{I.11}$$

where we used the KMS condition to push $\mathcal{O}_b$ around to the left of $\mathcal{O}_a$ in (I.10). We then split the t_b contour in (I.10) into two pieces, one purely Lorentzian integral from $t_b = 0$ to $t_b = s_c + t_c$ and another purely Euclidean integral from $t_b = s_c + t_c$ to $t_b = s_c + t_c + 2\pi i(n-1)$. Again, this is the full answer for the replica three point function, $\mathcal{A}_n^{(3)}$, at all n excluding the replica diagonal terms.

From this we can compute the leading order in n correction to the three-point function (dropping the diagonal terms). Taking an n-derivative and setting $n \to 1$, the total correction is

$$\begin{aligned}
\mathcal{A}_n^{(3)} \sim{}& \frac{i(n-1)}{2\pi}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{s_c} dt_b \frac{\langle [\mathcal{O}_b(-is_b - it_b), \mathcal{O}_a(0)]\mathcal{O}_c(-is_c + \tau_{ca})\rangle_1}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\
&+ (\text{replica diagonal terms}) + \mathcal{O}\left((n-1)^2\right).
\end{aligned} \tag{I.12}$$

Replica Diagonal Terms

For future reference, we now list the replica diagonal (or $j = k$) terms that we have suppressed. In the order we considered above, we have

$$\begin{aligned} &n\theta(\tau_{cb})\theta(\tau_{ba})\sum_{k=0}^{n-1}\langle\mathcal{O}_a(0)\mathcal{O}_b(2\pi k+\tau_{ba})\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n \\ &= n\theta(\tau_{cb})\theta(\tau_{ba})\Bigg(\langle\mathcal{O}_a(0)\mathcal{O}_b(\tau_{ba})\mathcal{O}_c(\tau_{ca})\rangle_n - \\ &\frac{1}{2\pi i}\int_{i2\pi}^{i2\pi n}dt_c\int_{-\infty}^{\infty}ds_c\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_c-it_c-\tau_{cb})\mathcal{O}_c(-is_c-it_c)\rangle_n}{4\sinh^2((s_c-i\tau_{ca})/2)}\Bigg). \end{aligned} \tag{I.13}$$

Again, other orderings can be found just by swapping the a, b, c labels accordingly. Note that at $n = 1$, the integral term vanishes and the answer reduces to the angular ordered three-point function as expected.

J Explicit Calculation of $c^{(2)}$

In this section, we compute the OPE coefficient of $\hat{T}_{++}$ in the $\hat{D}_+ \times \hat{D}_+$ OPE. This requires us to compute the twist defect three point function $\langle\Sigma_n^0\hat{D}_+\hat{D}_+\hat{T}_{--}\rangle$. As described around equation (H.3), the appearence of a delta function in the $\hat{D}_+ \times \hat{D}_+$ OPE requires that the coefficient c_n for $\hat{T}_{--}$ must be at least of order $(n-1)^2$ near $n = 1$. We now show that this is indeed true. In the next section, we will explicitly compute the anomalous dimension of $\hat{T}_{--}$ and show that it behaves as $g_n \sim \gamma^{(1)}(n-1) + \mathcal{O}((n-1)^2)$. We will finally show that their ratio obeys the relation

$$c^{(2)}/\gamma^{(1)} = 2\pi/S_{d-3} \tag{J.1}$$

as required by the first law of entanglement entropy.

The three point function we are after, at integer n, takes the form

$$\begin{aligned} &\langle\Sigma_n^0\hat{T}_{--}(y')\hat{D}_+(y)\hat{D}_+(y=0)\rangle \\ &= -\oint d\bar{z}\oint d\bar{w}\oint\frac{du}{2\pi iu}\langle\Sigma_n^0 T_{--}(u,\bar{u}=0,y')T_{++}(z=0,\bar{z},y)T_{++}(w=0,\bar{w},0)\rangle \end{aligned} \tag{J.2}$$

where it is understood that all the stress tensor operators should be $\mathbb{Z}_n$ symmetrized. Our goal is now to analytically continue this expression in n and then expand around $n = 1$. We can turn to the previous section for this result, letting $\mathcal{O}_a = T_{++}(w = 0, \bar{w}, 0)$, $\mathcal{O}_b = T_{++}(z = 0, \bar{z}, y)$ and $\mathcal{O}_c = T_{--}(u, \bar{u} = 0, 0)$.

Just as in Section 7.5, a major simplification occurs for this correlator; the two displacement operators are space-like separated from each other, so they commute even upon analytic continuation. Thus, any terms with commutators between $\mathcal{O}_a$ and $\mathcal{O}_b$ in the previous section drop out.

Furthermore, the so-called "replica diagonal" terms in the previous section will also vanish. This is because they do not contain enough s-integrals that produce necessary poles in $\bar{z}$ and $\bar{w}$. Thus, these terms vanish upon the contour integration over $\bar{z}$ and $\bar{w}$ in (J.2).

These considerations together with equation (I.11) of the previous section make it clear that the correlator in (J.2) vanishes up to order $(n-1)^2$. Indeed, the only surviving contribution is the second term in (I.11). Expanding that to second order while being careful to account for the spin of the stress tensors, we find

$$\langle \Sigma_n^0 \hat{T}_{--} \hat{D}_+ \hat{D}_+ \rangle_n = \\ \frac{-(n-1)^2}{2} \oint d\bar{z} d\bar{w} \frac{du}{2\pi i u} \int_0^\infty \int_0^\infty d\lambda_b d\lambda_c \lambda_b^2 \lambda_c^2 \frac{\langle T_{++}(\bar{z}\lambda_b, y) T_{++}(\bar{w}\lambda_c) T_{--}(u, y') \rangle}{(\lambda_b - 1 - i\epsilon)^2 (\lambda_c - 1 + i\epsilon)^2} + \mathcal{O}((n-1)^3). \tag{J.3}$$

Rescaling $\lambda_b \to \lambda_b / \bar{z}$ and $\lambda \to \lambda_c / \bar{w}$, we can then expand the denominators in small $\bar{z}, \bar{w}$ and perform the residue projections in $\bar{z}, \bar{w}$ and u. The final answer is the simple result

$$\langle \Sigma_n^0 \hat{T}_{--} \hat{D}_+ \hat{D}_+ \rangle = 2\pi^2 (n-1)^2 \langle \mathcal{E}_+(y) \mathcal{E}_+(y=0) T_{--}(u=0, y') \rangle + \mathcal{O}((n-1)^3). \tag{J.4}$$

where $\mathcal{E}_+(y)$ is the half-averaged null energy operator

$$\mathcal{E}_+(y) = \int_0^\infty d\lambda \, T_{++}(z=0, \lambda, y) \tag{J.5}$$

We now set about computing this correlator. Expanding the stress tensor three point function in a general CFT into the free field basis, we have

$$\langle TTT \rangle = n_s \langle TTT \rangle_s + n_f \langle TTT \rangle_f + n_v \langle TTT \rangle_v \tag{J.6}$$

where n_s, n_f and n_v are charges characterizing the specific theory.

One can demonstrate that the only non-vanishing contribution from these three terms is from the scalar three point function. The way to see this is as follows. The fermion term can be computed by considering a putative free Dirac fermion theory with field ψ. The stress tensor looks like $T_{++} \sim \bar{\psi} \Gamma_+ \partial_+ \psi$. Then we can compute the $\langle TTT \rangle$ three point function via Wick contractions. There will always be at least one Wick contraction between operators in each T_{++}. The kinematics of these operators ensure that such a contraction vanishes because they are both on the same null plane.[7]

The same argument can be made for the vector fields. In fact, the *only* reason that the scalar contribution doesn't vanish is because of the presence of a total derivative term in the

[7]Actually these contractions will be proportional to a delta function $\delta^{d-2}(y)$ but we are assuming the three stress tensors sit at different y's.

conformal stress tensor, namely $T_{++} \supset -\frac{d-2}{4(d-1)}\partial_+^2 :\phi^2:$. One can then show that the only non-vanishing term is

$$\langle \mathcal{E}_+(y)\mathcal{E}_+(0)T_{--}(y')\rangle = \frac{4n_s(d-2)}{(d-1)^3}\frac{1}{|y|^{d-2}|y'|^{2d}}. \tag{J.7}$$

Dividing by the two point function $\langle T_{++}(0)T_{--}(y')\rangle = \frac{c_T}{4|y'|^{2d}}$, we find

$$c^{(2)} = \frac{32\pi^2 n_s(d-2)}{c_T(d-1)^3}. \tag{J.8}$$

We now turn to computing the anomalous dimension $\gamma^{(1)}$ for the stress tensor operator $\hat{T}$ on the defect.

K Explicit Calculation of $\gamma^{(1)}$

In this section, we will follow the steps laid out in [9] for computing the spectrum of defect operators and associated anomalous dimension induced by the bulk stress tensor. To do this, we must compute

$$n\sum_{j=0}^{n-1} \langle \Sigma_n^0 T_{--}(w,0,y)T_{++}(0,\bar{z},0)\rangle \,. \tag{K.1}$$

To leading order in $n-1$ this expression takes the form of a sum of two terms, a "modular energy" piece and a "relative entropy" piece

$$\begin{aligned}(\partial_n - 1)\langle \Sigma_n^0 \hat{T}_{--}\hat{T}_{++}\rangle|_{n=1} = \Big(&-2\pi\langle HT_{--}(w,0,y)T_{++}(0,\bar{z},0)\rangle \\ &- \int_0^{-\infty} d\lambda \frac{\lambda^2}{(\lambda-1+i\epsilon)^2}\langle T_{--}(w,0,y)T_{++}(0,\bar{z}\lambda,0)\rangle\Big)\end{aligned} \tag{K.2}$$

We will try to extract the anomalous dimensions and spectra of operators by examining the two point function of the defect stress tensor. In this framework, the signal of an anomalous dimension is a logarithmic divergence. As explained in [9], the log needs to be cutoff by $\bar{z}w/y^2$ or $z\bar{w}/y^2$. In fact, there will be two such logarithms that will add to make the final answer single-valued on the Euclidean section.

We are thus tasked with looking for all of the terms containing log divergences in (K.2). Since the modular Hamiltonian is just a local integral of the stress tensor

$$H = \int d^{d-2}y' \int_0^\infty dx^+ x^+ T_{++}(x^- = 0, x^+, y') \tag{K.3}$$

then the first term on the r.h.s. of (K.2) is a stress tensor three point function. Following the method of the previous section, we can then break up (K.2) into the free field basis.

This determines both terms on the r.h.s of (K.2) in terms of charges n_s, n_f and n_v. This allows us to instead compute the answer in a theory of free massless scalars, fermions and vectors. While this might seem like three times the work, it actually illuminates why g_n is only dependent on n_s. We start by examining the case of a free scalar and will see why the free fermion and free vector terms do not contribute to g_n.

Spectrum induced by free scalar

This spectrum of $\phi(z, \bar{z}, y)$ was analyzed in [13]. The authors found that the leading twist defect primaries are all twist one (in $d = 4$) and have dimension independent of n. As noted in Appendix C of that work, this can be understood in any dimension from the fact that ϕ is annihilated by the bulk Laplacian. This constraint - for defect primaries - enforces holomorphicity in $z, \bar{z}$ of the bulk-defect OPE which translates to a lack of anomalous dimensions. For free fermions and vectors, the same argument goes through since their two point functions are also annihilated by the Laplacian.

One might be confused because the anomalous dimension for scalar operators of dimension Δ was computed in [9] and found to be non-zero for operators of dimension $\Delta = \frac{d-2}{2}$. This discrepancy has to do with a subtlety related to the extra boundary term in the modular Hamiltonian for free scalars. This discrepancy is related to the choice of the stress tensor - the traceless, conformal stress tensor vs. the canonical stress tensor.

The authors of [13] worked with *canonical* free fields, for which the stress tensor is just $T_{++}^{\text{canonical}} = \partial_+\phi\partial_+\phi$. Indeed if one inserts the canonical stress tensor into the modular Hamiltonian in equation (3.20) of [9], then the anomalous dimension vanishes. On the other hand, if one uses the conformal stress tensor, $T_{++}^{\text{conformal}} = :\partial_+\phi\partial_+\phi: - \frac{(d-2)}{4(d-1)}\partial_+^2 :\phi^2:$, then anomalous dimension for ϕ is given by [9].

This discrepancy thus amounts to a choice of the stress tensor. Note that this is special to free scalars and does not exist for free fermions and vectors since there are no dimension $d-2$ scalar primaries in these CFTs. This proves that if one works with canonical free fields, there should be no anomalous dimension for the defect operators induced by the fundamental fields ϕ, ψ and A_μ. This is enough to prove that the defect primary induced by the *canonical* bulk stress tensor must also have zero anomalous dimension since this is just formed by normal-ordered products of the defect primaries induced by the bulk fundamental fields.

Back to the stress tensor

The upshot is that we only need to worry about the terms in (K.2) proportional to n_s. Furthermore, we only need to worry about terms in the $\langle HTT\rangle$ term that involve the boundary term of the modular Hamiltonian. This reduces the expression down to the term

$$\langle HTT\rangle \supset -\frac{(d-2)}{4(d-1)}\int d^{d-2}y\, \langle :\phi^2: T_{++}(0,\bar{z},y)T_{--}(w,0,0)\rangle\,. \tag{K.4}$$

A simple calculation shows that the only contractions that give log divergences come from

$$\begin{aligned}\langle HTT\rangle \supset& \frac{n_s(d-2)^2}{4(d-1)^2}\int d^{d-2}y'\,\langle\phi(0,0,y')\phi(0,0,0)\rangle\,\langle\phi(0,0,y')\partial_{\bar z}^2\phi(0,\bar z,0)T_{--}(0,0,y)\rangle\\ =& -\frac{n_s c_{\phi\phi}^3 d(d-2)^4}{16(d-1)^3}\int d^{d-2}y'\frac{1}{|y'|^{d-2}|y-y'|^{d-2}|y|^{d+2}}.\end{aligned} \tag{K.5}$$

This integral has two log divergences coming from $y'=0$ and $y'=y$, however they can be regulated by fixing $z,\bar z$ and $w,\bar w$ away from zero. The two singularities just add to make the final answer single valued under rotations by 2π about the defect as in [9]. We thus find

$$\langle HTT\rangle \supset -n_s\frac{c_{\phi\phi}^3 d(d-2)^4}{32(d-1)^3}S_{d-3}\log(w\bar w z\bar z/|y|^4)\frac{1}{|y|^{2d}} = -\frac{2n_s(d-2)}{(d-1)^3}S_{d-3}\log(w\bar w z\bar z/|y|^4)\frac{1}{|y|^{2d}}. \tag{K.6}$$

Dividing by $\langle T_{++}T_{--}\rangle$ gives

$$\gamma^{(1)} = \frac{16\pi n_s(d-2)}{c_T(d-1)^3}S_{d-3}. \tag{K.7}$$

Comparing with (J.8), we see that

$$\frac{c^{(2)}}{\gamma^{(1)}} = \frac{2\pi}{S_{d-3}} \tag{K.8}$$

as required by the first law of entanglement.

L Calculating $\mathcal{F}_n$

At first glance, $\mathcal{F}_n$ seems difficult to calculate; we would like a method to compute this correlation function at leading order in $n-1$ without having to analytically continue a $\mathbb{Z}_n$ symmetrized four point function. The method for analytic continuation is detailed in Appendix I.

As detailed in Appendix I, part of what makes the analytic continuation in n difficult is the analytic structure (branch cuts) due to various operators becoming null separated from each other in Lorentzian signature. One might naively worry that we have to track this for four operators in the four point function $\mathcal{F}_n$.

We will leverage the fact that the two stress tensors in $\hat D_+(y_1)$ and $\hat D_+(y_2)$ are in the lightcone limit with respect to the defect since

$$\hat D_+(y_1) = \lim_{|z|\to 0} i\oint d\bar z \sum_{j=0}^{n-1} T_{++}^{(j)}(z=0,\bar z,y_1). \tag{L.1}$$

Thus, the stress tensors at y_1 and y_2 commute with each other even after a finite amount of boost. This means that these two operators do not see each other in the analytic continuation. In other words, the analytic structure for each of these operators is just that of a $\mathbb{Z}_n$ symmetrised *three* point function. This was computed in Appendix I.

We can thus jump straight to (I.12) but now with two $\mathcal{O}_b$ operators. The final replica four point function assuming $[\mathcal{O}_{b_1}, \mathcal{O}_{b_2}] = 0$ is given by[8]

$$\frac{(n-1)}{8\pi^2}\int_{-\infty}^{\infty} ds_c ds_{b_1} ds_{b_2} \int_0^{s_c} dt_{b_1} dt_{b_2} \frac{\langle [\mathcal{O}_{b_2}(-is_{b_2}-it_{b_2}), [\mathcal{O}_{b_1}(-is_{b_1}-it_{b_1}), \mathcal{O}_a(0)]]\mathcal{O}_c(-is_c+\tau_{ca})\rangle_1}{64\sinh^2((s_{b_1}-i\tau_{b_1a})\sinh^2((s_{b_2}-i\tau_{b_2a})/2)\sinh^2((s_c-i\epsilon)/2)}$$
$$+\,\mathcal{O}((n-1)^2). \tag{L.2}$$

To make contact with $\mathcal{F}_n$, we assign

$$\begin{aligned}
\mathcal{O}_{b_1}(-is_1) &= \lim_{|z|\to 0} i \oint d\bar{z}\, e^{2s_1-2i\tau_{b_1a}} T_{++}(x^- = 0, x^+ = r_{\bar{z}}e^{s_1}, y_1)\\
\mathcal{O}_{b_2}(-is_2) &= \lim_{|w|\to 0} i \oint d\bar{w}\, e^{2s_2-2i\tau_{b_2a}} T_{++}(x^- = 0, x^+ = r_{\bar{w}}e^{s_2}, y_2)\\
\mathcal{O}_c(-is_c) &= \lim_{|u|\to 0} i \oint du\, e^{-2s_c+2i\tau_{ca}} T_{--}(x^- = -r_u e^{-s_c}, x^+ = 0, y_4)\\
\mathcal{O}_a(0) &= \lim_{|v|\to 0} i \oint \frac{dv}{2\pi i} T_{--}(x^- = -r_v, x^+ = 0, y_3)
\end{aligned} \tag{L.3}$$

with $\bar{z}, \bar{w} = r_{\bar{z},\bar{w}}e^{i\tau_{b_1,b_2}}$ and $u, v = r_{u,v}e^{-i\tau_{a,c}}$. The funny factors of $e^{2s-2i\tau}$ are to account for the spin of the stress tensor.

Shifting $s_{b_{1,2}} \to s_{b_{1,2}} - t_{b_{1,2}} - \log(r_{1,2})$ and moving to null coordinates $\lambda = e^s$, we find the expression

$$\begin{aligned}
&\mathcal{F}_n = \lim_{|z|,|w|,|u|,|v|\to 0} \oint d\bar{z}\, d\bar{w}\, du\, dv\, \times\\
&\frac{(n-1)}{8\pi^2}\int_{-\infty}^{\infty} ds_c \int_0^{\infty} \frac{d\lambda_{b_{1,2}}\, \lambda_{b_1}^2 \lambda_{b_2}^2}{\bar{z}^3\bar{w}^3} \int_0^{s_c} dt_{b_1} dt_{b_2} e^{-s_c} e^{-t_{b_1}-t_{b_2}} e^{6i\tau_a} \,\times\\
&\frac{\langle [T_{++}(x^+ = \lambda_{b_1}), [T_{++}(x^+ = \lambda_{b_2}), T_{--}(x^- = -r_v)]]T_{--}(x^- = -r_u e^{-s_c - i\tau_{ca}})\rangle_1}{\left(\frac{\lambda_{b_1}e^{i\tau_a}}{\bar{z}e^{t_{b_1}}} - 1\right)^2 \left(\frac{\lambda_{b_2}e^{i\tau_a}}{\bar{w}e^{t_{b_2}}} - 1\right)^2 (e^{s_c - i\epsilon} - 1)^2}.
\end{aligned} \tag{L.4}$$

The first line in (L.4) comes from the residue projections in the definitions of the displacement operators. Expanding the integrand at small $|\bar{z}|$ and $|\bar{w}|$, we can perform the residue integrals over $\bar{z}$ and $\bar{w}$ leaving us with

[8] We have dropped the so-called "replica diagonal" terms in (I.12) since they will drop out of the final answer after the residue projection in (L.1).

$$\mathcal{F}_n = \lim_{|u|,|v|\to 0} \oint du\, dv \times$$
$$\frac{1-n}{2}\int_{-\infty}^{\infty} ds_c \int_0^{s_c} dt_{b_1} dt_{b_2} e^{-s_c+2i\tau_a} e^{t_{b_1}+t_{b_2}} \frac{\langle [\mathcal{E}_+(y_1), [\mathcal{E}_+(y_2), T_{--}(x^- = -r_v)]] T_{--}(x^- = -ue^{-s_c+i\tau_a})\rangle_1}{(e^{s_c-i\epsilon}-1)^2} \tag{L.5}$$

where $\mathcal{E}_+(y_1)$ is a half-averaged null energy operator, $\int_0^\infty dx^+ T_{++}(x^+)$.

We can now do the t_{b_1} and t_{b_2} integrals which produce two factors of $e^{s_c}-1$ precisely cancelling the denominator. Note that a similar cancellation occurred in equation (7.6.12). We can then replace commutators of half-averaged null energy operators with commutators of full averaged null energy operators. Using the fact that $\hat{\mathcal{E}}_+ |\Omega\rangle = 0$, we are left with the expression

$$\mathcal{F}_n = \lim_{|v|,|u|\to 0} \oint dudv \times$$
$$\frac{(1-n)}{2}\int_{-\infty}^{\infty} ds_c\, e^{-s_c+2i\tau_a} \left\langle T_{--}(x^- = -r_v, x^+ = 0, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^- = -ue^{-s_c+i\tau_a}, x^+ = 0, y_4)\right\rangle_1 . \tag{L.6}$$

Using boost invariance, we can also write this as

$$\mathcal{F}_n = 4\pi^2(n-1)\int_{-\infty}^{\infty} ds_c\, e^{-s_c} \left\langle T_{--}(x^- = -1, x^+ = 0, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^- = -e^{-s_c}, x^+ = 0, y_4)\right\rangle_1 \tag{L.7}$$

where we have performed the projection over v, u.

This is precisely the formula we were after. From here, one can just insert the $\hat{\mathcal{E}}_+ \times \hat{\mathcal{E}}_+$ OPE as described in the main text.

M Free Field Theories and Null Quantization

In this section we review the basics of null quantization (see [139, 26]). We then show that our computations in Section 7.6 can reproduce the results of [26]. In free (and super-renormalizable) quantum field theories, one can evolve the algebra of operators on some space-like slice up to the null plane $x_- = 0$ and quantize using the null generator $P_+ = \int d^{d-2}y\; dx^+\; T_{++}(x^+, y)$ as the Hamiltonian. One can show that for free scalar fields, the algebra on the null plane factorizes across each null-generator (or "pencil") of the $x^- = 0$ plane. For each pencil, the algebra $\mathcal{A}_{p_y}$ is just the algebra associated to a 1+1-d chiral CFT.

Accordingly, the vacuum state factorizes as an infinite tensor product of $1+1$-d chiral CFT vacua:

$$|\Omega\rangle = \bigotimes_y |\Omega\rangle^{p_y} \tag{M.1}$$

where $|0\rangle_{p_y}$ is the vacuum for the chiral $1+1$-d CFT living on the pencil at transverse coordinate y.

Thus, if we trace out everything to the past of some (possibly wiggly) cut of the null plane defined by $x^+ = X^+(y)$, we will be left with an infinite product of reduced vacuum density matrices for a $1+1$-d CFT on the pencil

$$\sigma_{X^+(y)} = \bigotimes_y \sigma^{p_y}_{x^+ > X^+(y)}. \tag{M.2}$$

As discussed in [26], a general excited state on the null plane $|\Psi\rangle$ can also be expanded in the small transverse size of $\mathcal{A}$ of a given pencil. For any p_y, the full reduced density matrix above some cut of the null plane takes the form

$$\rho = \sigma^{p_y}_{X^+(y)} \otimes \rho^{(0)}_{\text{aux}} + \mathcal{A}^{1/2} \sum_{ij} \sigma^{p_y}_{X^+(y)} \int dr d\theta f_{ij}(r,\theta) \partial\phi(re^{i\theta}) \otimes E_{ij}(\theta) \tag{M.3}$$

where $\partial\phi$ is an operator acting on the pencil Hilbert space and $E_{ij}(\theta) = e^{\theta(K_i - K_j)} |i\rangle \langle j|$, with $|i\rangle$ eigenvectors for the auxiliary modular Hamiltonian, K_{aux}. Note that E_{ij} parameterizes our ignorance about the rest of the state on the null plane which is not necessarily the vacuum.

As a consistency check of (7.6.12), we now demonstrate agreement with the result of [26]. In null quantization, the delta function piece of the shape deformation corresponds to a shape deformation of the pencil while keeping the auxiliary system fixed. Note that the ansatz M.3 is analogous to the λ expansion in Section 7.6 even though we are now considering a general excited state

$$\rho = \sigma + \mathcal{A}^{1/2} \delta\rho + \mathcal{O}(\mathcal{A}). \tag{M.4}$$

We now just plug in our expression of $\delta\rho$ into (7.6.8) and find that the relative entropy second variation is

$$\begin{aligned}\frac{d^2}{dX^+(y)^2} S_{\text{rel}}(\rho|\rho_0) = \frac{1}{2} \sum_{ij} \int \int (drd\theta)_1 (drd\theta)_2 (f_{ij}(r,\theta))_1 (f_{ji}(r,\theta))_2 \\ \int ds\ e^s \langle (\partial\phi)_1 \mathcal{E}_+ \mathcal{E}_+ (\partial\phi)_2(s) \rangle_{\text{p}} \langle E_{ij}(\theta_1) E_{ji}(\theta_2 - is) \rangle_{\text{aux}}.\end{aligned} \tag{M.5}$$

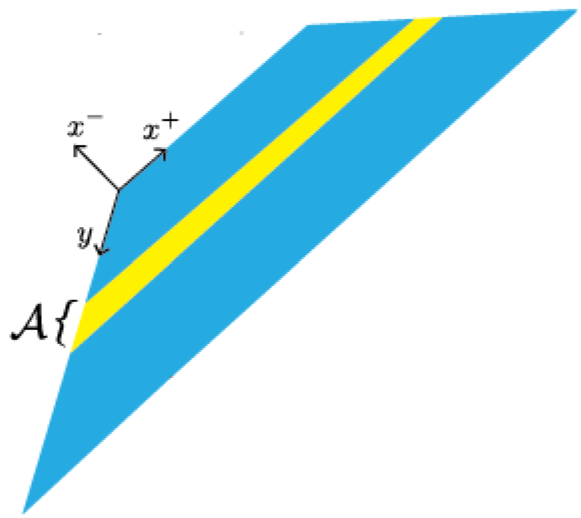

Figure 8.3: The Hilbert space on a null hypersurface of a free (or superrenormalizable) quantum field theory factorizes across narrow pencils of width $\mathcal{A}$. One pencil is shown above in yellow. The neighboring pencils then can be thought of as an auxiliary system (shown in blue). In the vacuum, the state between the pencil and the auxiliary system factorizes, but in an excited state there could be nontrivial entanglement between the two systems.

Now on the pencil, $\mathcal{E}_+$ is the translation generator so we can use the commutator $i[\mathcal{E}_+, \partial\phi] = \partial^2\phi$ and the fact that $\mathcal{E}_+ |0\rangle = 0$ to get

$$\frac{d^2}{dX^+(y)^2} S_{\text{rel}}(\rho|\rho_0) = \frac{1}{2}\sum_{ij}\int\int (drd\theta)_1 (drd\theta)_2 (f_{ij}(r,\theta))_1 (f_{ji}(r,\theta))_2$$
$$\int dse^s \langle(\partial^3\phi)_1(\partial\phi)_2(s)\rangle_{\text{p}} \langle E_{ij}(\theta_1)E_{ji}(\theta_2 - is)\rangle_{\text{aux}}. \tag{M.6}$$

Using the chiral two-point function we have

$$\langle(\partial^3\phi)_1(\partial\phi)_2(s)\rangle_{\text{p}} = \frac{e^s}{(r_1 e^{i\theta_1} - r_2 e^{i\theta_2 + s})^4}. \tag{M.7}$$

Moreover, the auxiliary correlator is given by

$$\langle E_{ij}(\theta_1)E_{ji}(\theta_2 - is)\rangle = e^{-2\pi K_i} e^{\nu_{ij}(\theta_1-\theta_2+is)}, \ \nu_{ij} = K_i - K_j \tag{M.8}$$

We now shift the integration contour by $s \to s + i(\theta_1 - \theta_2) + i\pi + \log(r_1/r_2)$. Putting this all together we are left with evaluating

$$e^{-\pi(K_i+K_j)} e^{-2i(\theta_1+\theta_2)} \left(\frac{r_1}{r_2}\right)^{i\nu_{ij}} \frac{1}{(r_1r_2)^2} \int_{-\infty}^{\infty} ds \frac{e^{is\nu_{ij}} e^{2s}}{(1+e^s)^4}. \tag{M.9}$$

The θ integrals project us onto the $m = 2$ Fourier modes of f_{ij}, $f_{ij}^{(m=2)}(r)$, and we find the final answer

$$\frac{d^2}{dX^+(y)^2} S_{\rm rel}(\rho|\rho_0) = \frac{1}{2}\sum_{ij} |F_{ij}^{(2)}|^2 e^{-\pi(K_i+K_j)} g(\nu_{ij}) \tag{M.10}$$

where

$$F_{ij}^{(m)} = \int \frac{dr}{r^m} r^{i\nu_{ij}} f_{ij}^{(m)}(r), \;\; g(\nu) = \frac{\pi\nu(1+\nu^2)}{\sinh(\pi\nu)}. \tag{M.11}$$

This is precisely the answer that was found by different methods in [26]. Note that the right hand side of (M.10) is manifestly positive as required by the QNEC.